EXTREMAL RAYS OF THE CONE OF BETTI TABLES OF COMPLETE INTERSECTIONS

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December 15, 2015

THE BASICS

WHAT IN TARNATION IS A BETTI DIAGRAM?!

Modules

Betti diagrams

Polynomial ring in n variables: $k[x, y, z]$ $(n = 3)$

Mod out by pure powers ideal: $I = (x^a, y^b, z^c)$

WLOG $a \leq b \leq c$

Complete intersection: $\dfrac{k[x, y, z]}{I}$

WHAT IS $\beta_{i,j}$?

$\beta_{0,0} = 1$

$\beta_{1,j} = \#$ of generators of the ideal of degree j

$\beta_{i,j} = \#$ collections of i generators where degrees sum to j

EXAMPLE!

$$\beta(2,3,3) := \beta\left(\frac{k[x,y,z]}{(x^2,y^3,z^3)}\right)$$

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Remember $\beta_{1,j} = \#$ of generators of the ideal of degree j

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$$\begin{bmatrix} 1 & - & \beta_{2,2} & \beta_{3,3} \\ - & 1 & \beta_{2,3} & \beta_{3,4} \\ - & 2 & \beta_{2,4} & \beta_{3,5} \\ - & - & \beta_{2,5} & \beta_{3,6} \\ - & - & \beta_{2,6} & \beta_{3,7} \\ - & - & \beta_{2,7} & \beta_{3,8} \end{bmatrix}$$

$\beta_{2,3} = \#$ of pairs of generators of the ideal whose degrees sum to 3

EXAMPLE!

$$\beta(2,3,3) := \beta\left(\frac{k[x,y,z]}{(x^2,y^3,z^3)}\right)$$

$$\begin{bmatrix} 1 & - & - & \beta_{3,3} \\ - & 1 & - & \beta_{3,4} \\ - & 2 & - & \beta_{3,5} \\ - & - & 2 & \beta_{3,6} \\ - & - & 1 & \beta_{3,7} \\ - & - & - & \beta_{3,8} \end{bmatrix}$$

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Embed Betti diagrams in vector space

$$V = \bigoplus_{i \in \mathbb{Z}_{\geq 0}} \bigoplus_{j \in \mathbb{Z}} \mathbb{Q}$$

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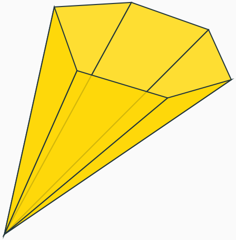
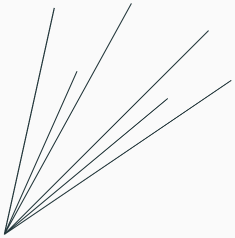
Consider the cone

$$B = \left\{ \sum_{k=1}^{l} r_k \beta^{(k)}(a_1^{(k)}, \cdots, a_{n_k}^{(k)}) \;\middle|\; r_k \in \mathbb{Q}_{\geq 0},\ l \in \mathbb{Z}_{\geq 0} \right\}$$



RATIONAL CONES



Find extremal rays of

$$B = \left\{ \sum_{k=1}^{l} r_k \beta^{(k)}(a_1^{(k)}, \cdots, a_{n_k}^{(k)}) \,\middle|\, r_k \in \mathbb{Q}_{\geq 0},\ l \in \mathbb{Z}_{\geq 0} \right\}$$

Come up with a partial order on these extremal rays

THE THEOREM

Conjecture: The $\beta^{(k)}(a_1^{(k)}, \cdots, a_{n_k}^{(k)})$ form the set of extremal rays for the cone $B = \left\{ \sum_{k=1}^{l} r_k \beta^{(k)}(a_1^{(k)}, \cdots, a_{n_k}^{(k)}) \mid r_k \in \mathbb{Q}_{\geq 0},\ l \in \mathbb{Z}_{\geq 0} \right\}$

Show $\beta(a_1, \cdots, a_c) \neq \sum_{k=1}^{l} r_k \beta^{(k)}(a_1^{(k)}, \cdots, a_{n_k}^{(k)})$ for distinct $\beta^{(k)}$

Simply, ${\beta^{(k)}}_{0,0} = 1$ for each k.

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$$\begin{bmatrix} 1 & - & - & - \\ - & 1 & - & - \\ - & 2 & - & - \\ - & - & 2 & - \\ - & - & 1 & - \\ - & - & - & 1 \end{bmatrix}$$

For each $\beta^{(k)}$, the bottom right 1 lands in the same place:

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1. For each $\beta^{(k)}(a_1^{(k)}, \cdots, a_{n_k}^{(k)})$, $n_k = n$

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1. For each $\beta^{(k)}(a_1^{(k)}, \cdots, a_{n_k}^{(k)})$, $n_k = n$
2. For each k, $a_1^{(k)} + \cdots + a_{n_k}^{(k)} = a_1 + \cdots + a_n$

Take $\beta(2,3,3)$ and $\beta(2,3)$:

REDUCTION ONE EXAMPLE

Take $\beta(2,3,3)$ and $\beta(2,3)$:

$$\begin{bmatrix} 1 & - & - & - \\ - & 1 & - & - \\ - & 2 & - & - \\ - & - & 2 & - \\ - & - & 1 & - \\ - & - & - & 1 \end{bmatrix}$$

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$$\begin{bmatrix} 1 & - & - & - \\ - & 1 & - & - \\ - & 2 & - & - \\ - & - & 2 & - \\ - & - & 1 & - \\ - & - & - & {\color{orange}1} \end{bmatrix} + \begin{bmatrix} 1 & - & - & - \\ - & 1 & - & - \\ - & 1 & 1 & - \\ - & - & {\color{orange}1} & - \\ - & - & - & - \\ - & - & - & - \end{bmatrix}$$

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$$\begin{bmatrix} 1 & - & - & - \\ - & 1 & - & - \\ - & 2 & - & - \\ - & - & 2 & - \\ - & - & 1 & - \\ - & - & - & {\color{orange}1} \\ - & - & - & - \end{bmatrix}$$

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REDUCTIONS OVERVIEW

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2. Degrees of generators of complete intersections sum to the same number.

PROOF

WHERE ARE WE NOW?

Main theorem proved for $n = 2, n = 3$.

It remains to induct on n.

Given a Betti Table β, define $-\beta$ by:

$$(-\beta)_{i,j} := -(\beta_{i,j})$$

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As an example, consider $\beta = \beta(2,3,3)$. Then,

$$\beta = \begin{bmatrix} 1 & - & - & - \\ - & 1 & - & - \\ - & 2 & - & - \\ - & - & 2 & - \\ - & - & 1 & - \\ - & - & - & 1 \end{bmatrix} \text{ and } -\beta = \begin{bmatrix} (-1) & - & - & - \\ - & (-1) & - & - \\ - & (-2) & - & - \\ - & - & (-2) & - \\ - & - & (-1) & - \\ - & - & - & (-1) \end{bmatrix}$$

Let G $:= \{\beta \mid \beta$ is a Betti Table$\} \cup \{-\beta \mid \beta$ is a Betti Table$\}$

Proposition

G is an abelian group under $+$

Recall that for G to be an abelian group, G has to:

1. have closure under $+$
2. be associative
3. have an identity element
4. have inverses, and
5. be commutative

Note that we can also create a multiplicative structure, $\odot$, on G.

Consider the closure of G under $\odot$; call it R.

Theorem

R is an integral domain under $+, \odot$

For R to be an integral domain, R has to:

1. be a commutative ring with unity under $+, \odot$, so:
   1.1 Abelian group under $+$
   1.2 Closure of $\overline{\mathrm{B}}$ under $\odot$
   1.3 Associative under $\odot$
   1.4 Commutative under $\odot$
   1.5 Distributivity of $\odot$ over $+$, and
   1.6 Identity element under $\odot$
2. have no zero divisors

WHY DOES IT MATTER?

Since R lacks zero divisors, multiplication is cancellative:
If $\alpha, \beta, \omega \in$ G and $\omega \neq 0$. Then, $\alpha \odot \omega = \beta \odot \omega \implies \alpha = \beta$

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Recall, we are interested in $\beta(a_1, \ldots, a_n) = \sum_{k=1}^{l} r_k \beta^{(k)}(a_1^{(k)}, \cdots, a_n^{(k)})$

Furthermore:
Fact: $\beta(a_1, \ldots, a_n) = \beta(a_1, \ldots, a_{n-1}) \odot \beta(a_n)$

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Therefore, if ${a_n}^{(k)} = a_n$ for all k, we can reduce to the $n - 1$ case, as $\odot$ is distributive and cancellative

1. Conjecture is a theorem for $n = 2$ and $n = 3$
2. We have an inductive step that puts us closer to the general case

WHAT'S UP NEXT

1. Finish induction on n for main theorem
2. Identify a partial order for diagrams in B
3. Use partial order to obtain structural results about the cone (for example, what else is in the cone?)

Questions?