Wreath Products in Algebraic Voting Theory

Committed to Committees

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Outline

Algebraic Framework

- Ballot Structure
- Vote Aggregation
- Point Allocation (Voting System)
- Election Results

Ballots

Ballots → List of allowable tabloids

$$\begin{array}{|c|c|c|}\hline A & B & C \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline A & B \\ \hline C & \multicolumn{1}{|c}{} \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|}\cline{1-1} A & \multicolumn{1}{|c}{} \\ \hline B & C \\ \hline \end{array} \qquad \begin{array}{|c|}\hline A \\ \hline B \\ \hline C \\ \hline \end{array}$$

These are elements of $\chi^{(3)}$, $\chi^{(2,1)}$, $\chi^{(1,2)}$, and $\chi^{(1,1,1)}$, respectively.

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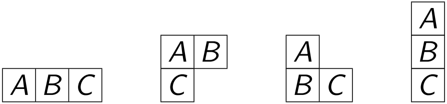

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Many elections focus on a single type of ballot structure. An exception to this is approval voting.

Vote Aggregation: The Profile Space

The components of the profile vector indicate the number of votes for a given tabloid option on the ballot.

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For an election in which voters give full rankings of three candidates (i.e. an election on the tabloids of $\chi^{(1,1,1)}$) the following is an example of a profile:

$$\vec{p} = \begin{array}{cl} \begin{pmatrix} 3 \\ 2 \\ 0 \\ 2 \\ 0 \\ 4 \end{pmatrix} & \begin{array}{l} ABC \\ ACB \\ BAC \\ BCA \\ CAB \\ CBA \end{array} \end{array}$$

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Definition

An *FG-module* is a vector space over the field F where there is a representation of the group G on the vector space, and multiplication is understood as the associated group action.

$$\begin{pmatrix}3\\2\\0\\2\\0\\4\end{pmatrix}\begin{array}{l}ABC\\ACB\\BAC\\BCA\\CAB\\CBA\end{array} \rightarrow (12) \rightarrow \begin{pmatrix}0\\2\\3\\2\\4\\0\end{pmatrix}\begin{array}{l}ABC\\ACB\\BAC\\BCA\\CAB\\CBA\end{array}$$

Point Allocation: Voting Systems

Consider the following two voting systems for a full ranking $(\chi^{(1,1,1)})$

- Plurality → $(1, 0, 0)$

$$\begin{pmatrix}3\\2\\0\\2\\0\\4\end{pmatrix}\begin{array}{c}ABC\\ACB\\BAC\\BCA\\CAB\\CBA\end{array} \to \begin{pmatrix}5\\2\\4\end{pmatrix}\begin{array}{c}A\\B\\C\end{array}$$

- Borda → $(2, 1, 0)$

Point Allocation: Voting Systems

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Voting Systems as Linear Transformations

The previous Borda example can be represented as the following linear transformation:

$$T_{(2,1,0)} = \begin{array}{cc} & \begin{array}{cccccc} ABC & ACB & BAC & BCA & CAB & CBA \end{array} \\ & \left(\begin{array}{cccccc} 2 & 2 & 1 & 0 & 1 & 0 \\ 1 & 0 & 2 & 2 & 0 & 1 \\ 0 & 1 & 0 & 1 & 2 & 2 \end{array}\right) \begin{array}{c} A \\ B \\ C \end{array} \end{array}$$

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Any system that gives candidates points based on their position within voters' rankings can be represented by a similar linear transformation.

Results Space

When one of these voting systems is applied to a profile, a results vector is created.

A results vector encodes the number of points that every potential outcome receives for the election.

$$T_w(\vec{p}) = \vec{r}$$

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The results spaces are also $\mathbb{Q}S_n$-module.

Schur's Lemma

Both profile spaces and results spaces are $\mathbb{Q}S_n$-modules. These positional scoring systems are $\mathbb{Q}S_n$-module homomorphisms, which can be proven with the previous fact and the fact that these systems are neutral.

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Definition

A submodule is a subspace of a module that is invariant under the group action.

Key Points of Algebraic Voting Theory

Domain

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Domain $\rightarrow$ Maps

Key Points of Algebraic Voting Theory

Domain → Maps → Codomain

Voting for Committees

What happens if we want to elect a subset of the candidates rather than a single candidate?

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What happens if we want to elect a subset of the candidates rather than a single candidate?

$$\begin{pmatrix}4\\2\\0\\2\\1\\5\\3\end{pmatrix}\begin{matrix}A\\B\\C\\AB\\AC\\BC\\ABC\end{matrix}\rightarrow$$

Voting for Committees

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$$\begin{pmatrix} 4 \\ 2 \\ 0 \\ 2 \\ 1 \\ 5 \\ 3 \end{pmatrix} \begin{array}{c} A \\ B \\ C \\ AB \\ AC \\ BC \\ ABC \end{array} \rightarrow \begin{pmatrix} \\ \\ \\ \\ \\ \\ \\ \end{pmatrix} \begin{array}{c} A \\ B \\ C \\ AB \\ AC \\ BC \\ ABC \end{array}$$

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Basketball team

Consider a basketball team of 15 players where three people specialize in each of the 5 positions: Point Guard, Shooting Guard, Small Forward, Power Forward, and Center. Now we want to vote for the best team.

- Structural limitation on how people can vote
- Create maps for specific purposes
- Need something more than just permuting names of "candidates."...

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- Need something more than just permuting names of "candidates."... A wreath product is what we need.

Wreath Products

Let G be a group and S_n be the symmetric group on the set $N = \{1, 2, \ldots, n\}$ of n elements.

The **wreath product** of G by S_n, denoted $G[S_n]$, is the semidirect product $G^n \rtimes S_n$ where $G[S_n] = \{(f; \pi) : f \in G^n,\ \pi \in S_n\}$

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In general, $G^n = \{f : N \to G\}$

Wreath Product for $G = S_m$

In the case of $G = S_m$, it is possible to express $S_m[S_n]$ in the following way:

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$$S_m[S_n] = \{(\sigma_1, \ldots, \sigma_n; \pi) : \sigma_i \in S_m, \pi \in S_n\}$$

Wreath Products

Some situations where wreath products are applicable:

Basketball Team, $S_3[S_5]$

- 15 players per team
- 5 positions, 3 players per position

2 Question 3 Response Referendum, $S_3[S_2]$

- 2 proposals
- 3 possible answers: Yes, No, Obstain

Action of $S_m[S_n]$

Given the element structure $(\sigma_1, \ldots, \sigma_n; \pi)$ we can now concisely define multiplication in $S_m[S_n]$:

$$(\sigma_1, \sigma_2, \ldots, \sigma_n;\ \pi)(\delta_1, \delta_2, \ldots, \delta_n;\ \tau)$$

$$= (\sigma_1\delta_{\pi^{-1}(1)}, \sigma_2\delta_{\pi^{-1}(2)}, \ldots, \sigma_n\delta_{\pi^{-1}(n)};\ \pi\tau)$$

for $\sigma_i, \delta_i \in S_m$ and $\pi, \tau \in S_n$

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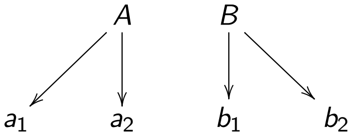
Example: Selecting committees with $S_2[S_2]$

Suppose we have two departments A and B, each with two members a_1, a_2 and b_1, b_2, respectively.



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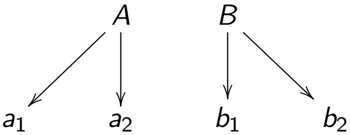
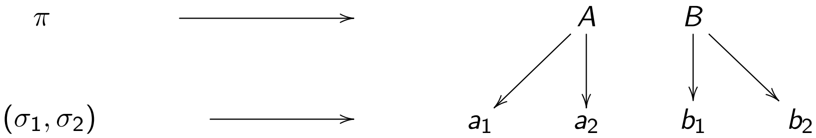


There are 4 possible committees:
$W = \{a_1, b_1\}, X = \{a_1, b_2\}, Y = \{a_2, b_1\}$ and $Z = \{a_2, b_2\}$

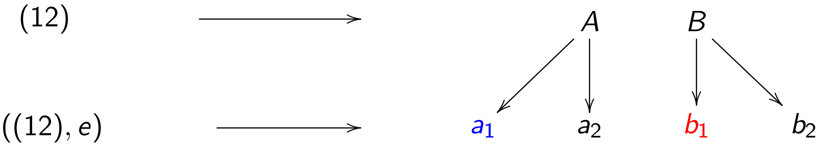
Example: Selecting committes with $S_2[S_2]$

First, how does $(\sigma_1, \sigma_2;\ \pi) \in S_2[S_2]$ act on a single committee?



Example: Selecting committees with $S_2[S_2]$

Consider the element $\varphi = ((12), e; (12))$ acting on committee $W = \{a_1, b_1\}$



(12)

$((12), e)$

A B

a_1 a_2 b_1 b_2

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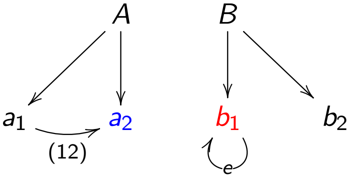

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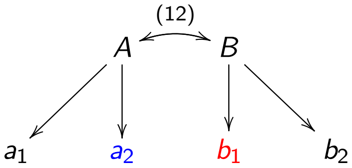

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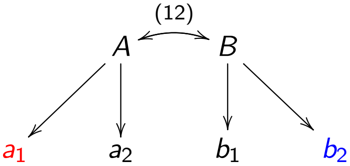

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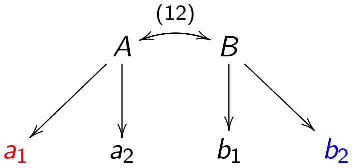

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$\varphi(W) = X$

Group Action on the Profile and Results Space

We can now define the group action of the wreath product on the profile and results space, P and R, respectively.

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$$\vec{p} = \begin{array}{c c} \begin{bmatrix} 1 \\ 0 \\ 3 \\ 2 \end{bmatrix} & \begin{array}{c} a_1 b_1 \\ a_1 b_2 \\ a_2 b_1 \\ a_2 b_2 \end{array} \end{array} = \begin{array}{c c} \begin{bmatrix} 1 \\ 0 \\ 3 \\ 2 \end{bmatrix} & \begin{array}{c} W \\ X \\ Y \\ Z \end{array} \end{array}$$

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Let's apply $\varphi = ((12), e;\ (12))$ as before:

$$\varphi(\begin{bmatrix} 1 \\ 0 \\ 3 \\ 2 \end{bmatrix}) \begin{matrix} W \\ X \\ Y \\ Z \end{matrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 3 \end{bmatrix} \begin{matrix} X \\ Z \\ W \\ Y \end{matrix}$$

Group Action on the Profile and Results Space

Another way of viewing the action of the wreath product:

$$\begin{bmatrix} 1 \\ 0 \\ 3 \\ 2 \end{bmatrix} \begin{array}{l} a_1 b_1 \\ a_1 b_2 \\ a_2 b_1 \\ a_2 b_2 \end{array}$$

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$$\varphi\left(\begin{bmatrix}1\\0\\3\\2\end{bmatrix}\right)\begin{matrix}a_1b_1\\a_1b_2\\a_2b_1\\a_2b_2\end{matrix} = ((12), e;\ (12))\begin{bmatrix}1\\0\\3\\2\end{bmatrix}\begin{matrix}a_1b_1\\a_1b_2\\a_2b_1\\a_2b_2\end{matrix}$$

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$$\varphi(\begin{bmatrix}1\\0\\3\\2\end{bmatrix})\begin{matrix}W\\X\\Y\\Z\end{matrix} = \begin{bmatrix}0\\2\\1\\3\end{bmatrix}\begin{matrix}X\\Z\\Y\\W\end{matrix}$$

$\mathbb{Q}S_m[S_n]$ Modules

Defining the action of $S_m[S_n]$ on P and R allows us to view them as FG-modules; specifically $\mathbb{Q}(S_m[S_n])$-modules. In order to apply Schur's Lemma, all the remains is to show that a voting procedure $T : P \to R$ is a $\mathbb{Q}S_m[S_n]$-module homomorphism, then decompose each space into its respective simple submodules.

Decomposing a $\mathbb{Q}S_m[S_n]$-module

We will now outline the process of decomposing a $\mathbb{Q}S_m[S_n]$-module into submodules. We will use algorithms by Rockmore (1995) and James and Liebeck (2001).

Submodules of a $\mathbb{Q}S_m[S_n]$-module are indexed by tuples of partitions which add up to n.

$$S_3[S_5]$$

$$S^{\left(\square\square,\,\square,\,\begin{smallmatrix}\square\\\square\end{smallmatrix}\right)}$$

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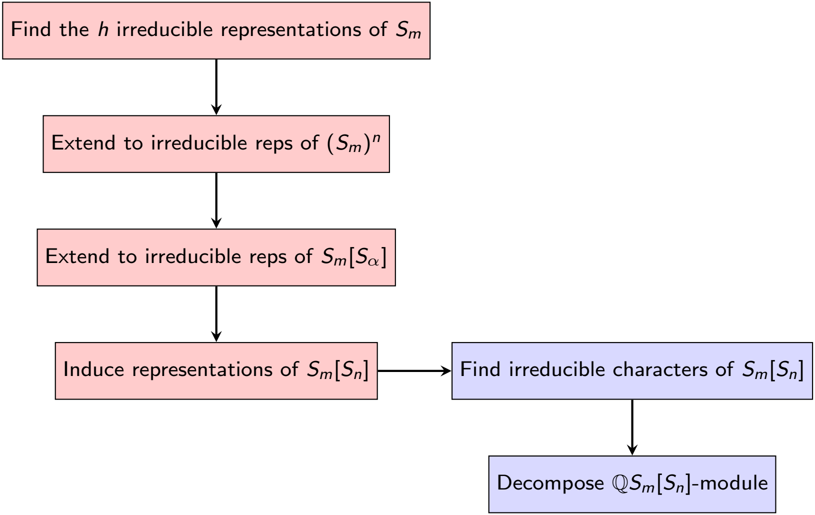
$$S^{(\ydiagram{2,2},\emptyset,\ydiagram{1})}$$

Decomposing a $\mathbb{Q}S_m[S_n]$-module



Find the h irreducible representations of S_m

Extend to irreducible reps of $(S_m)^n$

Extend to irreducible reps of $S_m[S_\alpha]$

Induce representations of $S_m[S_n]$

Find irreducible characters of $S_m[S_n]$

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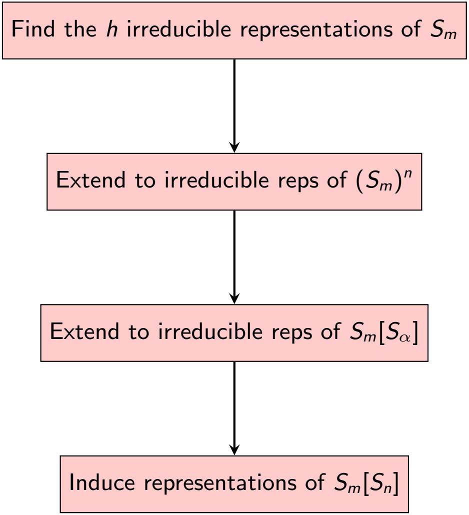

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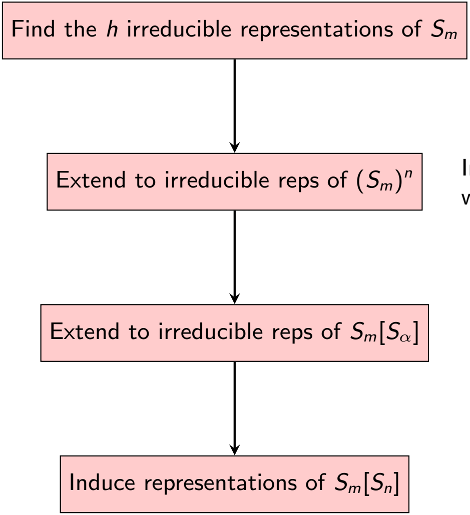


Indexed by weak compositions of n with h parts

$$S_3[S_5]$$

$$m = 3 = h, n = 5$$

$$\alpha = (3, 0, 2)$$

Decomposing a $\mathbb{Q}S_m[S_n]$-module

Find the h irreducible representations of S_m

Extend to irreducible reps of $(S_m)^n$

$$\alpha = (3, 0, 2)$$

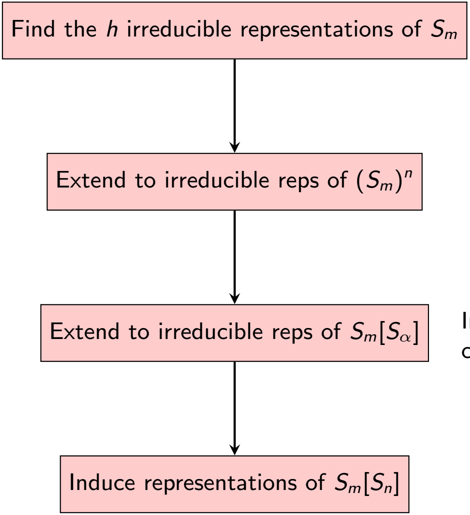
Extend to irreducible reps of $S_m[S_\alpha]$

Indexed by h-tuples of partitions corresponding to α

$$\lambda_1 = (\ydiagram{2,1}, \emptyset, \ydiagram{2})$$

Induce representations of $S_m[S_n]$

Decomposing a $\mathbb{Q}S_m[S_n]$-module



Find the h irreducible representations of S_m

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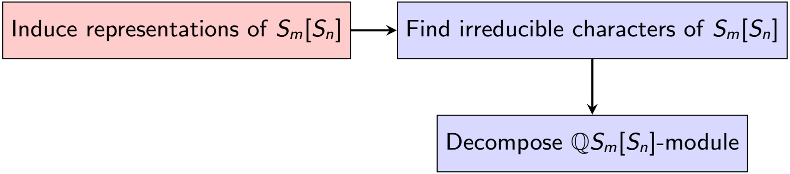
Induce representations of $S_m[S_n]$

$$\alpha = (3, 0, 2)$$

Indexed by h-tuples of partitions corresponding to α

$$\lambda_2 = (\ydiagram{3}, \emptyset, \ydiagram{1,1})$$

Decomposing a $\mathbb{Q}S_m[S_n]$-module



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Find irreducible characters of $S_m[S_n]$

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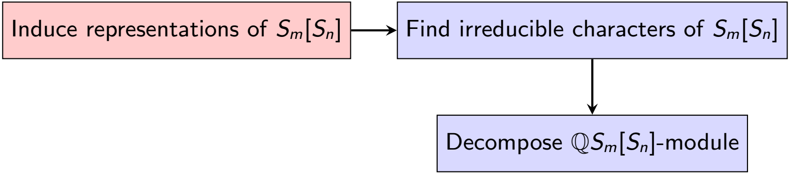

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Decompose $\mathbb{Q}S_m[S_n]$-module

1. Determine a basis $v_1, \ldots, v_k$ for the $\mathbb{Q}S_m[S_n]$-module vector space.
2. For each irreducible character of $S_m[S_n]$ calculate

$$v_i \left(\sum_{g \in S_m[S_n]} \chi(g^{-1})g \right)$$

for each basis vector v_i. The resulting vectors span the submodule S^χ.

Example: The $S_2[S_2]$ case

We will do the decomposition for the $S_2[S_2]$ case. Our $\mathbb{Q}S_2[S_2]$-module is 4-dimensional and indexed by each of the possible pairs of candidates. Here is a basis:

$$\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix} \begin{matrix}a_1b_1\\a_1b_2\\a_2b_1\\a_2b_2\end{matrix}$$

Example: The $S_2[S_2]$ case

	e	$(e,(12);e)$	$((12),(12);e)$	$(e,e;(12))$	$(e,(12);(12))$
$(\square\square, \emptyset)$	1	1	1	1	1
$(\begin{smallmatrix}\square\\\square\end{smallmatrix}, \emptyset)$	1	1	1	-1	-1
$(\square, \square)$	2	0	-2	0	0
$(\emptyset, \square\square)$	1	-1	1	1	-1
$(\emptyset, \begin{smallmatrix}\square\\\square\end{smallmatrix})$	1	-1	1	-1	1

Example: The $S_2[S_2]$ case

	e	$(e,(12);e)$	$((12),(12);e)$	$(e,e;(12))$	$(e,(12);(12))$
(□, □)	2	0	-2	0	0

$$v_1\left(\sum_{g\in S_m[S_n]}\chi(g^{-1})g\right)=\begin{bmatrix}1\\0\\0\\0\end{bmatrix}(2e-2((12),(12);e)$$

$$=\begin{bmatrix}2\\0\\0\\0\end{bmatrix}e-\begin{bmatrix}2\\0\\0\\0\end{bmatrix}\begin{matrix}a_1b_1\\a_1b_2\\a_2b_1\\a_2b_2\end{matrix}((12),(12);e)$$

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	e	$(e,(12);e)$	$((12),(12);e)$	$(e,e;(12))$	$(e,(12);(12))$
$(\square,\square)$	2	0	-2	0	0

$$v_1\left(\sum_{g\in S_m[S_n]}\chi(g^{-1})g\right) = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}(2e-2((12),(12);e)$$

$$= \begin{bmatrix}2\\0\\0\\0\end{bmatrix} - \begin{array}{cl}\begin{bmatrix}0\\0\\0\\2\end{bmatrix} & \begin{array}{l}a_1b_1\\a_1b_2\\a_2b_1\\a_2b_2\end{array}\end{array}$$

Example: The $S_2[S_2]$ case

	e	$(e,(12);e)$	$((12),(12);e)$	$(e,e;(12))$	$(e,(12);(12))$
$(\square,\square)$	2	0	-2	0	0

$$
\begin{aligned}
v_1\left(\sum_{g\in S_m[S_n]}\chi(g^{-1})g\right) &= \begin{bmatrix}1\\0\\0\\0\end{bmatrix}(2e-2((12),(12);e)\\
&= \begin{bmatrix}2\\0\\0\\0\end{bmatrix}-\begin{bmatrix}0\\0\\0\\2\end{bmatrix}=\begin{array}{c}\begin{bmatrix}2\\0\\0\\-2\end{bmatrix}\end{array}\begin{array}{l}a_1b_1\\a_1b_2\\a_2b_1\\a_2b_2\end{array}
\end{aligned}
$$

Example: The $S_2[S_2]$ case

We can now apply this same process to each of the other basis vectors v_i of our space V, and we obtain the following submodule:

$$S^{(\square,\square)} = \langle \begin{bmatrix} 2 \\ 0 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 0 \\ 2 \end{bmatrix} \rangle = \langle \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} \rangle$$

Example: The $S_2[S_2]$ case

$$S^{(\square\square,\emptyset)} = \left\langle \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\rangle \qquad S^{(\emptyset,\square\square)} = \left\langle \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \right\rangle$$

$$S^{\left(\begin{smallmatrix}\square\\\square\end{smallmatrix},\emptyset\right)} = \vec{0} \qquad S^{\left(\emptyset,\begin{smallmatrix}\square\\\square\end{smallmatrix}\right)} = \vec{0}$$

$$S^{(\square,\square)} = \left\langle \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} \right\rangle$$

Decompositions of $S_2[S_n]$

Conjecture (Lee, 2010 Thesis)

For $S_2[S_n]$ with $n \geq 2$, the results space decomposes into exactly $\bigoplus_\lambda S^\lambda$, the direct sum of irreducible submodules indexed by double trivial partitions $\lambda = (\lambda_1, \lambda_2)$ (the "flat" partitions).

Example: The $\mathbb{Q}S_3[S_2]$ case

As the wreath product $S_3[S_5]$ is not manageable by hand, we decided to decompose a corresponding $\mathbb{Q}S_3[S_2]$-module instead and see if the "flat partition" property holds.

$$\begin{array}{ccc}
S^{(\ydiagram{2},\emptyset,\emptyset)} & S^{(\ydiagram{1,1},\emptyset,\emptyset)} & S^{(\ydiagram{1},\ydiagram{1},\emptyset)} \\
S^{(\emptyset,\ydiagram{2},\emptyset)} & S^{(\emptyset,\ydiagram{1,1},\emptyset)} & S^{(\emptyset,\ydiagram{1},\ydiagram{1})} \\
S^{(\emptyset,\emptyset,\ydiagram{2})} & S^{(\emptyset,\emptyset,\ydiagram{1,1})} & S^{(\ydiagram{1},\emptyset,\ydiagram{1})}
\end{array}$$

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$$\begin{array}{ccc}
S^{(\square\square,\emptyset,\emptyset)} & S^{(\begin{smallmatrix}\square\\\square\end{smallmatrix},\emptyset,\emptyset)} & S^{(\square,\square,\emptyset)} \\
S^{(\emptyset,\square\square,\emptyset)} & S^{(\emptyset,\begin{smallmatrix}\square\\\square\end{smallmatrix},\emptyset)} & S^{(\emptyset,\square,\square)} \\
S^{(\emptyset,\emptyset,\square\square)} & S^{(\emptyset,\emptyset,\begin{smallmatrix}\square\\\square\end{smallmatrix})} & S^{(\square,\emptyset,\square)}
\end{array}$$

Example: The $\mathbb{Q}S_3[S_2]$ case

As the wreath product $S_3[S_5]$ is not manageable by hand, we decided to decompose a corresponding $\mathbb{Q}S_3[S_2]$-module instead and see if the “flat partition” property holds.

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S^{(\emptyset,\ydiagram{2},\emptyset)} & S^{(\emptyset,\ydiagram{1,1},\emptyset)} & S^{(\emptyset,\ydiagram{1},\ydiagram{1})} \\
S^{(\emptyset,\emptyset,\ydiagram{2})} & S^{(\emptyset,\emptyset,\ydiagram{1,1})} & S^{(\ydiagram{1},\emptyset,\ydiagram{1})}
\end{array}$$

Decompositions of $S_m[S_n]$

Conjecture (Calaway, Csapo, Samelson, 2015)

For $S_m[S_n]$ with $m, n \geq 2$, the results space decomposes into a direct sum composed only of irreducible submodules indexed by h-tuple trivial partitions (the "flat" partitions).

Going Forward

- Now that we understand how to decompose these spaces, apply these tools to the voting schemes we discussed previously (especially in the case of $S_3[S_5]$).
- Further investigate the decomposition of $S_3[S_n]$ and $S_m[S_n]$. In particular, can we rederive/generalize a 2011 result of Caselli and Fulci?
- Characterize more fully the separability of simple submodules.

Acknowledgements

- Batman
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The universe is an enormous direct product of representations of symmetry groups.

-Steven Weinberg

Bibliography I

[1] S. J. Brams and P. C. Fishburn. *Approval voting*. Birkhäuser, Boston, Mass., 1983.

[2] F. Caselli and R. Fulci. Refined Gelfand models for wreath products. *European J. Combin.*, 32(2):198–216, 2011.

[3] K.-D. Crisman. The Borda count, the Kemeny rule, and the permutahedron. In *The mathematics of decisions, elections, and games*, volume 624 of *Contemp. Math.*, pages 101–134. Amer. Math. Soc., Providence, RI, 2014.

[4] Z. Daugherty. An algebraic approach to voting theory. 2005. Senior Thesis at Harvey Mudd College.

[5] Z. Daugherty, A. K. Eustis, G. Minton, and M. E. Orrison. Voting, the symmetric group, and representation theory. *Amer. Math. Monthly*, 116(8):667–687, 2009.

[6] P. C. Fishburn and A. Pekec. Approval voting for committees: Threshold approaches. 2004. Available at http://people.duke.edu/ pekec/Publications/CommitteeVotePekecFishburn.pdf.

Bibliography II

[7] G. James and M. Liebeck. *Representations and characters of groups*. Cambridge University Press, New York, second edition, 2001.

[8] S. C. Lee. Understanding voting for committees using wreath products. 2010. Senior Thesis at Harvey Mudd College.

[9] T. C. Ratliff. Some startling inconsistencies when electing committees. *Soc. Choice Welf.*, 21(3):433–454, 2003.

[10] D. N. Rockmore. Fast Fourier transforms for wreath products. *Appl. Comput. Harmon. Anal.*, 2(3):279–292, 1995.

[11] B. E. Sagan. *The symmetric group*, volume 203 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 2001. Representations, combinatorial algorithms, and symmetric functions.