

Wreath Products in Algebraic Voting Theory

Committed to Committees

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Outline

- 1 Algebraic Framework
- 2 Voting for Committees
- 3 Wreath Products
- 4 Decomposing a $\mathbb{Q}S_m[S_n]$ -module

Algebraic Framework

- Ballot Structure
- Vote Aggregation
- Point Allocation (Voting System)
- Election Results

Ballots

Ballots \rightarrow List of allowable tabloids



These are elements of $\chi^{(3)}$, $\chi^{(2,1)}$, $\chi^{(1,2)}$, and $\chi^{(1,1,1)}$, respectively.

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Many elections focus on a single type of ballot structure. An exception to this is approval voting.

Vote Aggregation: The Profile Space

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For an election in which voters give full rankings of three candidates (i.e. an election on the tabloids of $\chi^{(1,1,1)}$) the following is an example of a profile:

$$\vec{p} = \begin{pmatrix} 3 \\ 2 \\ 0 \\ 2 \\ 0 \\ 4 \end{pmatrix} \begin{matrix} ABC \\ ACB \\ BAC \\ BCA \\ CAB \\ CBA \end{matrix}$$

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Definition

An *FG-module* is a vector space over the field F where there is a representation of the group G on the vector space, and multiplication is understood as the associated group action.

$$\begin{pmatrix} 3 \\ 2 \\ 0 \\ 2 \\ 0 \\ 4 \end{pmatrix} \begin{matrix} ABC \\ ACB \\ BAC \\ BCA \\ CAB \\ CBA \end{matrix} \rightarrow (12) \rightarrow \begin{pmatrix} 0 \\ 2 \\ 3 \\ 2 \\ 4 \\ 0 \end{pmatrix} \begin{matrix} ABC \\ ACB \\ BAC \\ BCA \\ CAB \\ CBA \end{matrix}$$

Point Allocation: Voting Systems

Consider the following two voting systems for a full ranking $(\chi^{(1,1,1)})$

- Plurality $\rightarrow (1, 0, 0)$

$$\begin{pmatrix} 3 \\ 2 \\ 0 \\ 2 \\ 0 \\ 4 \end{pmatrix} \begin{matrix} ABC \\ ACB \\ BAC \\ BCA \\ CAB \\ CBA \end{matrix} \rightarrow \begin{pmatrix} 5 \\ 2 \\ 4 \end{pmatrix} \begin{matrix} A \\ B \\ C \end{matrix}$$

- Borda $\rightarrow (2, 1, 0)$

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$$\begin{pmatrix} 3 \\ 2 \\ 0 \\ 2 \\ 0 \\ 4 \end{pmatrix} \begin{matrix} ABC \\ ACB \\ BAC \\ BCA \\ CAB \\ CBA \end{matrix} \rightarrow \begin{pmatrix} 10 \\ 11 \\ 12 \end{pmatrix} \begin{matrix} A \\ B \\ C \end{matrix}$$

Voting Systems as Linear Transformations

The previous Borda example can be represented as the following linear transformation:

$$T_{(2,1,0)} = \begin{array}{cccccc} & ABC & ACB & BAC & BCA & CAB & CBA \\ \left(\begin{array}{cccccc} 2 & 2 & 1 & 0 & 1 & 0 \\ 1 & 0 & 2 & 2 & 0 & 1 \\ 0 & 1 & 0 & 1 & 2 & 2 \end{array} \right) & \begin{array}{l} A \\ B \\ C \end{array} \end{array}$$

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Any system that gives candidates points based on their position within voters' rankings can be represented by a similar linear transformation.

Results Space

When one of these voting systems is applied to a profile, a results vector is created.

A results vector encodes the number of points that every potential outcome receives for the election.

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The results spaces are also $\mathbb{Q}S_n$ -module.

Schur's Lemma

Both profile spaces and results spaces are $\mathbb{Q}S_n$ -modules. These positional scoring systems are $\mathbb{Q}S_n$ -module homomorphisms, which can be proven with the previous fact and the fact that these systems are neutral.

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Definition

A submodule is a subspace of a module that is invariant under the group action.

Key Points of Algebraic Voting Theory

Domain

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Domain \rightarrow Maps

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Domain \rightarrow Maps \rightarrow Codomain

Voting for Committees

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$$\begin{pmatrix} 4 \\ 2 \\ 0 \\ 2 \\ 1 \\ 5 \\ 3 \end{pmatrix} \begin{matrix} A \\ B \\ C \\ AB \\ AC \\ BC \\ ABC \end{matrix} \rightarrow$$

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Basketball team

Consider a basketball team of 15 players where three people specialize in each of the 5 positions: Point Guard, Shooting Guard, Small Forward, Power Forward, and Center. Now we want to vote for the best team.

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- Create maps for specific purposes
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- Need something more than just permuting names of “candidates.” ... A wreath product is what we need.

Wreath Products

Let G be a group and S_n be the symmetric group on the set $N = \{1, 2, \dots, n\}$ of n elements.

The **wreath product** of G by S_n , denoted $G[S_n]$, is the semidirect product $G^n \rtimes S_n$ where $G[S_n] = \{(f; \pi) : f \in G^n, \pi \in S_n\}$

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In general, $G^n = \{f : N \rightarrow G\}$

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$$S_m[S_n] = \{(\sigma_1, \dots, \sigma_n; \pi) : \sigma_i \in S_m, \pi \in S_n\}$$

Wreath Products

Some situations where wreath products are applicable:

Basketball Team, $S_3[S_5]$

- 15 players per team

- 5 positions, 3 players per position

2 Question 3 Response Referendum, $S_3[S_2]$

- 2 proposals

- 3 possible answers: Yes, No, Abstain

Action of $S_m[S_n]$

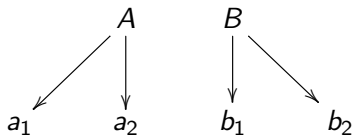
Given the element structure $(\sigma_1, \dots, \sigma_n; \pi)$ we can now concisely define multiplication in $S_m[S_n]$:

$$\begin{aligned} & (\sigma_1, \sigma_2, \dots, \sigma_n; \pi)(\delta_1, \delta_2, \dots, \delta_n; \tau) \\ &= (\sigma_1 \delta_{\pi^{-1}(1)}, \sigma_2 \delta_{\pi^{-1}(2)}, \dots, \sigma_n \delta_{\pi^{-1}(n)}; \pi\tau) \end{aligned}$$

for $\sigma_i, \delta_i \in S_m$ and $\pi, \tau \in S_n$

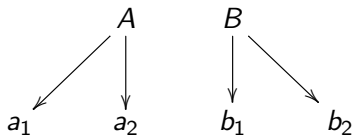
Example: Selecting committees with $S_2[S_2]$

Suppose we have two departments A and B , each with two members a_1, a_2 and b_1, b_2 , respectively.



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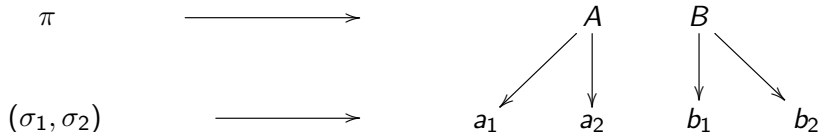


There are 4 possible committees:

$W = \{a_1, b_1\}$, $X = \{a_1, b_2\}$, $Y = \{a_2, b_1\}$ and $Z = \{a_2, b_2\}$

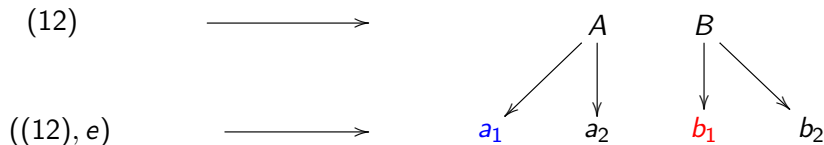
Example: Selecting committees with $S_2[S_2]$

First, how does $(\sigma_1, \sigma_2; \pi) \in S_2[S_2]$ act on a single committee?



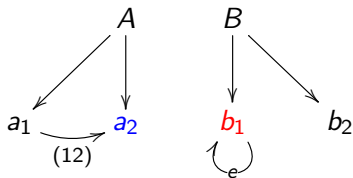
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Consider the element $\varphi = ((12), e; (12))$ acting on committee $W = \{a_1, b_1\}$



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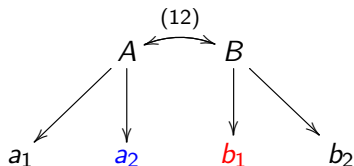
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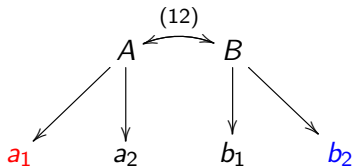
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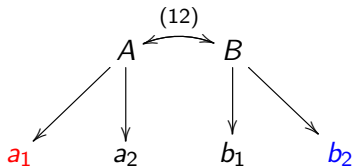


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$\varphi(W) = X$

Group Action on the Profile and Results Space

We can now define the group action of the wreath product on the profile and results space, P and R , respectively.

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$$\vec{p} = \begin{bmatrix} 1 \\ 0 \\ 3 \\ 2 \end{bmatrix} \begin{matrix} a_1 b_1 \\ a_1 b_2 \\ a_2 b_1 \\ a_2 b_2 \end{matrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \\ 2 \end{bmatrix} \begin{matrix} W \\ X \\ Y \\ Z \end{matrix}$$

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Let's apply $\varphi = ((12), e; (12))$ as before:

$$\varphi \left(\begin{bmatrix} 1 \\ 0 \\ 3 \\ 2 \end{bmatrix} \right) \begin{matrix} W \\ X \\ Y \\ Z \end{matrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 3 \end{bmatrix} \begin{matrix} X \\ Z \\ W \\ Y \end{matrix}$$

Group Action on the Profile and Results Space

Another way of viewing the action of the wreath product:

$$\begin{bmatrix} 1 \\ 0 \\ 3 \\ 2 \end{bmatrix} \begin{matrix} a_1 b_1 \\ a_1 b_2 \\ a_2 b_1 \\ a_2 b_2 \end{matrix}$$

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$$\varphi\left(\begin{bmatrix} 1 \\ 0 \\ 3 \\ 2 \end{bmatrix}\right) \begin{matrix} W \\ X \\ Y \\ Z \end{matrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 3 \end{bmatrix} \begin{matrix} X \\ Z \\ Y \\ W \end{matrix}$$

$\mathbb{Q}S_m[S_n]$ Modules

Defining the action of $S_m[S_n]$ on P and R allows us to view them as FG -modules; specifically $\mathbb{Q}(S_m[S_n])$ -modules. In order to apply Schur's Lemma, all that remains is to show that a voting procedure $T : P \rightarrow R$ is a $\mathbb{Q}S_m[S_n]$ -module homomorphism, then decompose each space into its respective simple submodules.

Decomposing a $\mathbb{Q}S_m[S_n]$ -module

We will now outline the process of decomposing a $\mathbb{Q}S_m[S_n]$ -module into submodules. We will use algorithms by Rockmore (1995) and James and Liebeck (2001).

Submodules of a $\mathbb{Q}S_m[S_n]$ -module are indexed by tuples of partitions which add up to n .

$$S_3[S_5]$$

$$S((\square\square, \square, \square))$$

Decomposing a $\mathbb{Q}S_m[S_n]$ -module

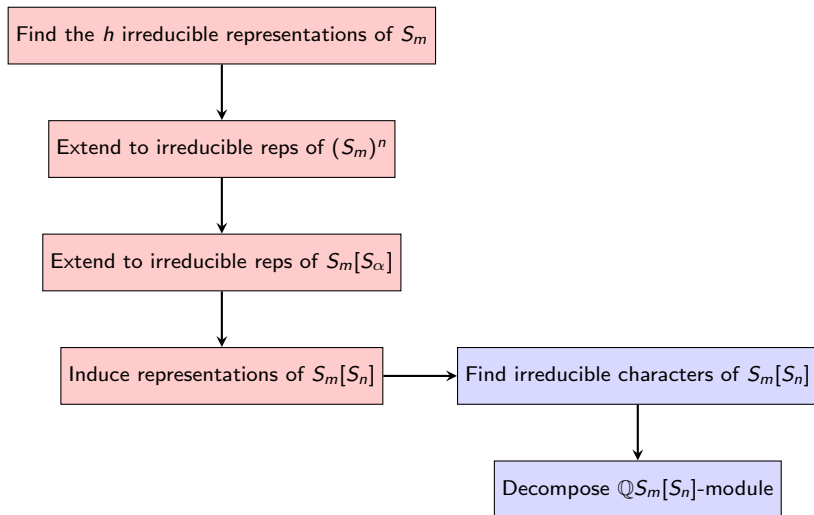
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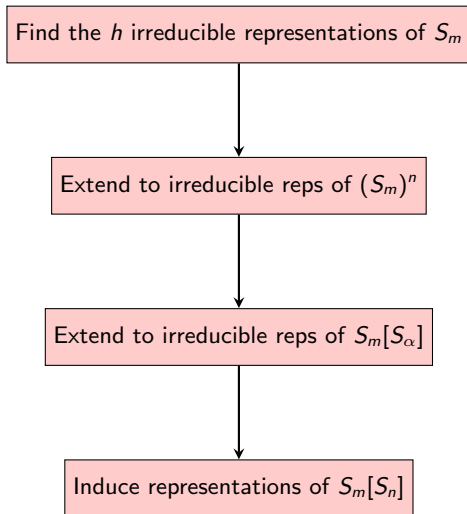
$$S_3[S_5]$$

$$S\left(\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}, \emptyset, \square\right)$$

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Decomposing a $\mathbb{Q}S_m[S_n]$ -module

Find the h irreducible representations of S_m



Extend to irreducible reps of $(S_m)^n$

Indexed by weak compositions of n
with h parts



Extend to irreducible reps of $S_m[S_\alpha]$

$$S_3[S_5]$$

$$m = 3 = h, n = 5$$

$$\alpha = (3, 0, 2)$$



Induce representations of $S_m[S_n]$

Decomposing a $\mathbb{Q}S_m[S_n]$ -module

Find the h irreducible representations of S_m



Extend to irreducible reps of $(S_m)^n$



Extend to irreducible reps of $S_m[S_\alpha]$



Induce representations of $S_m[S_n]$

$$\alpha = (3, 0, 2)$$

Indexed by h -tuples of partitions corresponding to α

$$\lambda_1 = \left(\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array}, \emptyset, \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} \right)$$

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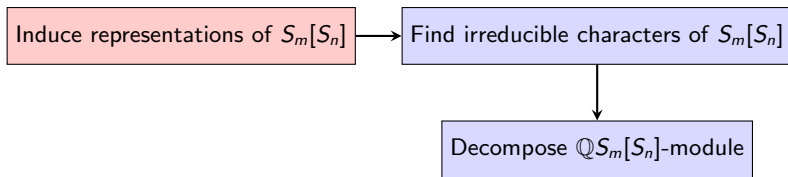
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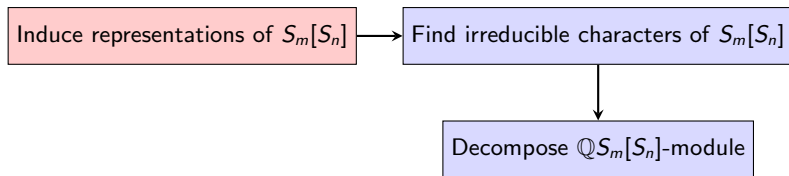
Indexed by h -tuples of partitions corresponding to α

$$\lambda_2 = (\square \square \square, \emptyset, \begin{array}{c} \square \\ \square \end{array})$$

Decomposing a $\mathbb{Q}S_m[S_n]$ -module



Decomposing a $\mathbb{Q}S_m[S_n]$ -module



- 1 Determine a basis v_1, \dots, v_k for the $\mathbb{Q}S_m[S_n]$ -module vector space.
- 2 For each irreducible character of $S_m[S_n]$ calculate

$$v_i \left(\sum_{g \in S_m[S_n]} \chi(g^{-1})g \right)$$

for each basis vector v_i . The resulting vectors span the submodule S^{χ} .

Example: The $S_2[S_2]$ case

We will do the decomposition for the $S_2[S_2]$ case. Our $\mathbb{Q}S_2[S_2]$ -module is 4-dimensional and indexed by each of the possible pairs of candidates. Here is a basis:

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \begin{matrix} a_1 b_1 \\ a_1 b_2 \\ a_2 b_1 \\ a_2 b_2 \end{matrix}$$

Example: The $S_2[S_2]$ case

	e	$(e, (12); e)$	$((12), (12); e)$	$(e, e; (12))$	$(e, (12); (12))$
$(\square \square, \emptyset)$	1	1	1	1	1
$(\begin{array}{c} \square \\ \square \end{array}, \emptyset)$	1	1	1	-1	-1
(\square, \square)	2	0	-2	0	0
$(\emptyset, \square \square)$	1	-1	1	1	-1
$(\emptyset, \begin{array}{c} \square \\ \square \end{array})$	1	-1	1	-1	1

Example: The $S_2[S_2]$ case

	e	$(e, (12); e)$	$((12), (12); e)$	$(e, e; (12))$	$(e, (12); (12))$
(\square, \square)	2	0	-2	0	0

$$\begin{aligned}
 v_1 \left(\sum_{g \in S_m[S_n]} \chi(g^{-1})g \right) &= \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} (2e - 2((12), (12); e)) \\
 &= \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} e - \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{matrix} a_1 b_1 \\ a_1 b_2 \\ a_2 b_1 \\ a_2 b_2 \end{matrix} ((12), (12); e)
 \end{aligned}$$

Example: The $S_2[S_2]$ case

	e	$(e, (12); e)$	$((12), (12); e)$	$(e, e; (12))$	$(e, (12); (12))$
(\square, \square)	2	0	-2	0	0

$$\begin{aligned}
 v_1 \left(\sum_{g \in S_m[S_n]} \chi(g^{-1})g \right) &= \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} (2e - 2((12), (12); e)) \\
 &= \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \end{bmatrix} \begin{matrix} a_1 b_1 \\ a_1 b_2 \\ a_2 b_1 \\ a_2 b_2 \end{matrix}
 \end{aligned}$$

Example: The $S_2[S_2]$ case

	e	$(e, (12); e)$	$((12), (12); e)$	$(e, e; (12))$	$(e, (12); (12))$
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 &= \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \\ -2 \end{bmatrix} \begin{matrix} a_1 b_1 \\ a_1 b_2 \\ a_2 b_1 \\ a_2 b_2 \end{matrix}
 \end{aligned}$$

Example: The $S_2[S_2]$ case

We can now apply this same process to each of the other basis vectors v_i of our space V , and we obtain the following submodule:

$$S^{(\square, \square)} = \left\langle \begin{bmatrix} 2 \\ 0 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 0 \\ 2 \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} \right\rangle$$

Example: The $S_2[S_2]$ case

$$S^{(\square\square, \emptyset)} = \left\langle \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\rangle$$

$$S^{(\emptyset, \square\square)} = \left\langle \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \right\rangle$$

$$S^{(\square, \emptyset)} = \vec{0}$$

$$S^{(\emptyset, \square)} = \vec{0}$$

$$S^{(\square, \square)} = \left\langle \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} \right\rangle$$

Decompositions of $S_2[S_n]$

Conjecture (Lee, 2010 Thesis)

For $S_2[S_n]$ with $n \geq 2$, the results space decomposes into exactly $\bigoplus_{\lambda} S^{\lambda}$, the direct sum of irreducible submodules indexed by double trivial partitions $\lambda = (\lambda_1, \lambda_2)$ (the “flat” partitions).

Example: The $\mathbb{Q}S_3[S_2]$ case

As the wreath product $S_3[S_2]$ is not manageable by hand, we decided to decompose a corresponding $\mathbb{Q}S_3[S_2]$ -module instead and see if the “flat partition” property holds.

$$\begin{array}{ccc}
 \mathcal{S}(\square\square, \emptyset, \emptyset) & \mathcal{S}(\begin{array}{c} \square \\ \square \end{array}, \emptyset, \emptyset) & \mathcal{S}(\square, \square, \emptyset) \\
 \mathcal{S}(\emptyset, \square\square, \emptyset) & \mathcal{S}(\emptyset, \begin{array}{c} \square \\ \square \end{array}, \emptyset) & \mathcal{S}(\emptyset, \square, \square) \\
 \mathcal{S}(\emptyset, \emptyset, \square\square) & \mathcal{S}(\emptyset, \emptyset, \begin{array}{c} \square \\ \square \end{array}) & \mathcal{S}(\square, \emptyset, \square)
 \end{array}$$

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As the wreath product $S_3[S_2]$ is not manageable by hand, we decided to decompose a corresponding $\mathbb{Q}S_3[S_2]$ -module instead and see if the “flat partition” property holds.

$$\begin{array}{ccc}
 \mathcal{S}(\square\square, \emptyset, \emptyset) & \mathcal{S}(\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array}, \emptyset, \emptyset) & \mathcal{S}(\square, \square, \emptyset) \\
 \mathcal{S}(\emptyset, \square\square, \emptyset) & \mathcal{S}(\emptyset, \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array}, \emptyset) & \mathcal{S}(\emptyset, \square, \square) \\
 \mathcal{S}(\emptyset, \emptyset, \square\square) & \mathcal{S}(\emptyset, \emptyset, \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array}) & \mathcal{S}(\square, \emptyset, \square)
 \end{array}$$

Example: The $\mathbb{Q}S_3[S_2]$ case

As the wreath product $S_3[S_2]$ is not manageable by hand, we decided to decompose a corresponding $\mathbb{Q}S_3[S_2]$ -module instead and see if the “flat partition” property holds.

$$\begin{array}{ccc}
 \mathcal{S}(\square\square, \emptyset, \emptyset) & \mathcal{S}(\begin{array}{c} \square \\ \square \end{array}, \emptyset, \emptyset) & \mathcal{S}(\square, \square, \emptyset) \\
 \mathcal{S}(\emptyset, \square\square, \emptyset) & \mathcal{S}(\emptyset, \begin{array}{c} \square \\ \square \end{array}, \emptyset) & \mathcal{S}(\emptyset, \square, \square) \\
 \mathcal{S}(\emptyset, \emptyset, \square\square) & \mathcal{S}(\emptyset, \emptyset, \begin{array}{c} \square \\ \square \end{array}) & \mathcal{S}(\square, \emptyset, \square)
 \end{array}$$

Decompositions of $S_m[S_n]$

Conjecture (Calaway, Csapo, Samelson, 2015)

For $S_m[S_n]$ with $m, n \geq 2$, the results space decomposes into a direct sum composed only of irreducible submodules indexed by h -tuple trivial partitions (the “flat” partitions).

Going Forward

- Now that we understand how to decompose these spaces, apply these tools to the voting schemes we discussed previously (especially in the case of $S_3[S_5]$).
- Further investigate the decomposition of $S_3[S_n]$ and $S_m[S_n]$. In particular, can we rederive/generalize a 2011 result of Caselli and Fulci?
- Characterize more fully the separability of simple submodules.

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The universe is an enormous direct product of representations of symmetry groups.

-Steven Weinberg

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