The Multiple-Urn Ehrenfest Model
A Look into Eigenanalysis and Hitting Times

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What will we cover today?
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(1) Markov chains
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(2) The Ehrenfest urn model
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2. The Ehrenfest urn model
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5. Hitting times
6. Summary
The Ehrenfest urn model example

Urn 0

3
1
4

Urn 1

M
2
What is a Markov chain?

A **Markov chain** is a sequence of random variables $X_0, X_1, X_2, \ldots$ such that

$$\Pr[X_{n+1} = s_{n+1} | X_0 = s_0, X_1 = s_1, \ldots, X_n = s_n] = \Pr[X_{n+1} = s_{n+1} | X_n = s_n]$$

where $s_0, s_1, \ldots, s_n, s_{n+1}$ are elements in the state space of the Markov chain.
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This is known as the **memoryless property**.
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\]
where $s_0, s_1, ..., s_n, s_{n+1}$ are elements in the state space of the Markov chain.

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Let $X_n$ denote the number of balls in Urn 1 at time $n = 0, 1, 2,...$. Then
$(X_0, X_1, ...)$ forms a Markov chain with the state space $S = \{0, 1, ..., M\}$. 

The Ehrenfest urn model example

![Diagram of the Ehrenfest urn model example with urns 0 and 1, showing the distribution of red (3) and blue (4) balls in urn 0, and red (M) and blue (2) balls in urn 1.]
Transition matrix

A $k \times k$ matrix $\mathbb{P}$ is said to be a **transition matrix** of a Markov chain $(X_0, X_1, ...)$ with state space $S = \{s_1, ..., s_k\}$ if

$$\Pr[X_{n+1} = s_j \mid X_n = s_i] = \mathbb{P}_{i,j}.$$

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For the Ehrenfest urn model with $M=5$ balls, the transition matrix is:

$$
\begin{bmatrix}
0 & 1 & 2 & 3 & 4 & 5 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1/5 & 0 & 4/5 & 0 & 0 & 0 & 0 \\
0 & 2/5 & 0 & 3/5 & 0 & 0 & 0 \\
0 & 0 & 3/5 & 0 & 2/5 & 0 & 0 \\
0 & 0 & 0 & 4/5 & 0 & 1/5 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0
\end{bmatrix}
$$
Stationary distribution

Let \((X_0, X_1, \ldots)\) be a Markov chain with state space \(S = \{s_1, \ldots, s_k\}\) and transition matrix \(P\). A row vector \(\pi = (\pi_1, \ldots, \pi_k)\) is said to be a stationary distribution for the Markov chain, if it satisfies

(i) \(\pi_i \geq 0\) for \(i = 1, \ldots, k\), and \(\sum_{i=1}^{k} \pi_i = 1\), and

(ii) \(\pi P = \pi\), meaning that \(\sum_{i=1}^{k} \pi_i P_{i,j} = \pi_j\) for \(j = 1, \ldots, k\).
The 3-urn Ehrenfest model

Urn 0

Urn 1

Urn 2
The $d$-urn Ehrenfest model
Motivation for studying the multiple-urn Ehrenfest model
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Real-world applications:
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(1) Population migration
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(3) Treatment allocation
Why is eigenanalysis important?
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(1) Eigenvalues can tell how a Markov chain behaves. For example, a finite Markov chain converges to a stationary distribution if and only if its transition matrix has eigenvalue 1 with multiplicity 1 and all other eigenvalues are of modulus less than 1.
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That is, we can find an invertible matrix $\mathbb{A}$ such that

$$\mathbb{P} = \mathbb{A}^{-1} \mathbb{D} \mathbb{A},$$

where $\mathbb{D}$ is a diagonal matrix with all the eigenvalues of $\mathbb{P}$ on its main diagonal.
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This helps us compute $\mathbb{P}^n$:

$$\mathbb{P}^n = \mathbb{A}^{-1} \mathbb{D}^n \mathbb{A}.$$
Eigenvalues of the 3-urn model

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$M = 3$: $\lambda = 1, \frac{1}{2}, \frac{1}{2}, 0, 0, 0, \frac{-1}{2}, \frac{-1}{2}, \frac{-1}{2}, \frac{-1}{2}$. 
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$M = 3$: $\lambda = 1, \frac{1}{2}, \frac{1}{2}, 0, 0, 0, \frac{-1}{2}, \frac{-1}{2}, \frac{-1}{2}.$

$M = 4$: $\lambda = 1, \frac{5}{8}, \frac{5}{8}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{-1}{8}, \frac{-1}{8}, \frac{-1}{8}, \frac{-1}{8}, \frac{-1}{2}, \frac{-1}{2}, \frac{-1}{2}, \frac{-1}{2}.$
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Our observation:
For the 3-urn model with $M$ balls, the transition matrix has $(M + 1)$ distinct eigenvalues equally distanced between 1 and $\frac{-1}{2}$, and the $k$th largest eigenvalue has multiplicity $k$. 
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Our observation:
For the 3-urn model with $M$ balls, the transition matrix has $(M + 1)$ distinct eigenvalues equally distanced between 1 and $-\frac{1}{2}$, and the $k$th largest eigenvalue has multiplicity $k$. Let $\alpha = \frac{3}{2^M}$. Then the eigenvalues are given by

$$1, (1 - \alpha)_2, (1 - 2\alpha)_3, ..., \left(\frac{-1}{2}\right)^{M+1}.$$
Mark Kac’s results

In 1947, mathematician Mark Kac examined the Ehrenfest urn model with 2 urns and found that the eigenvectors and eigenvalues of the transition matrix are determined by a system of linear equations using the generating function

\[ f(z) = \sum_{k=0}^{\infty} x_k z^k \]

where \( x_k \) is the \( k \)th component of the eigenvector \( x \).
Mark Kac’s results

Using Kac’s function as a model, we define the following polynomial to generate the row eigenvectors for the 3-urn model:

\[ f^M(z_1, z_2) = \sum_{s_1=0}^{M} \sum_{s_2=0}^{M-s_1} a(s_1, s_2) z_1^{s_1} z_2^{s_2} \]

where \( a(s_1, s_2) \) is the \((s_1, s_2)\)th component of the eigenvector \( \mathbf{a} \) from the transition matrix \( \mathbb{P} \).
Transition probabilities

The possible states for the 3-urn model are given by $s = (s_1, s_2)$. The transition probabilities are then
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$$\Pr[(s_1, s_2) \rightarrow (s_1 + 1, s_2)] = \frac{M - s_1 - s_2}{2M}$$

$$\Pr[(s_1, s_2) \rightarrow (s_1 + 1, s_2 - 1)] = \frac{s_2}{2M}$$

$$\Pr[(s_1, s_2) \rightarrow (s_1 - 1, s_2 + 1)] = \frac{s_1}{2M}$$

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$$\Pr[(s_1, s_2) \rightarrow (s_1, s_2 - 1)] = \frac{s_2}{2M}. $$
Derived partial differential equation

Using the fact that \( a \mathbb{P} = \lambda a \) for any eigenvector \( a \) with eigenvalue \( \lambda \), we are able to substitute in our transition probabilities, multiply each term by \( z_1^{s_1} z_2^{s_2} \), and sum over all of \( s_1 \) and \( s_2 \).
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After collecting like terms and simplifying, we obtain the partial differential equation

\[
(1 - z_1)(1 + z_1 + z_2) \frac{\partial f^M}{\partial z_1} + (1 - z_2)(1 + z_1 + z_2) \frac{\partial f^M}{\partial z_2} + M(z_1 + z_2 - 2\lambda)f^M(z_1, z_2) = 0.
\]
Generating function and eigenvalues for $d = 3$ urns

**Theorem**

The coefficients of the functions

$$f_{(r_1, r_2)}^M(z_1, z_2) = (1 - z_1)^{r_1}(1 - z_2)^{r_2}(1 + z_1 + z_2)^{r_3},$$

where $r_1, r_2, \text{ and } r_3$ are nonnegative integers and $r_1 + r_2 + r_3 = M$, define a set of linearly independent eigenvectors of the 3-urn $M$-ball transition matrix.
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where $r_1, r_2,$ and $r_3$ are nonnegative integers and $r_1 + r_2 + r_3 = M$, define a set of linearly independent eigenvectors of the 3-urn $M$-ball transition matrix.

**Corollary**

The eigenvalues of the 3-urn $M$-ball transition matrix are $\lambda = \frac{3}{2M}r_3 - \frac{1}{2}$ for $r_3 \in \{0, \cdots, M\}$. Eigenvalue $\lambda$ has multiplicity $\frac{2M}{3}(1 - \lambda) + 1$. 
Example: The row eigenvector matrix $A$ for $M = 3$ balls

Consider the function

$$f_{(r_1, r_2)}^3(z_1, z_2) = (1 - z_1)^{r_1}(1 - z_2)^{r_2}(1 + z_1 + z_2)^{r_3}.$$ 

For $r_1 + r_2 = \frac{2M}{3}(1 - \lambda)$, we have:
Example: The row eigenvector matrix $A$ for $M = 3$ balls

Consider the function $f^{3}_{(r_1,r_2)}(z_1,z_2) = (1 - z_1)^{r_1}(1 - z_2)^{r_2}(1 + z_1 + z_2)^{r_3}$. For $r_1 + r_2 = \frac{2M}{3}(1 - \lambda)$, we have:

$$0 = 0 + 0 = r_1 + r_2 = \frac{2M}{3}(1 - \lambda), \quad \lambda = 1$$
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- $0 = 0 + 0 = r_1 + r_2 = \frac{2M}{3}(1 - \lambda), \quad \lambda = 1$
- $1 = 1 + 0 = r_1 + r_2 = \frac{2M}{3}(1 - \lambda), \quad \lambda = \frac{1}{2}$
- $1 = 0 + 1 = r_1 + r_2 = \frac{2M}{3}(1 - \lambda), \quad \lambda = \frac{1}{2}$
Example: The row eigenvector matrix $A$ for $M = 3$ balls

Consider the function $f_{(r_1, r_2)}^3(z_1, z_2) = (1 - z_1)^r_1(1 - z_2)^r_2(1 + z_1 + z_2)^r_3$.

For $r_1 + r_2 = \frac{2M}{3}(1 - \lambda)$, we have:

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3. $1 = 0 + 1 = r_1 + r_2 = \frac{2M}{3}(1 - \lambda), \quad \lambda = \frac{1}{2}$
4. $2 = 2 + 0 = r_1 + r_2 = \frac{2M}{3}(1 - \lambda), \quad \lambda = 0$
5. $2 = 1 + 1 = r_1 + r_2 = \frac{2M}{3}(1 - \lambda), \quad \lambda = 0$
6. $2 = 0 + 2 = r_1 + r_2 = \frac{2M}{3}(1 - \lambda), \quad \lambda = 0$

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For $r_1 + r_2 = \frac{2M}{3}(1 - \lambda)$, we have:

\[
\begin{align*}
0 &= 0 + 0 = r_1 + r_2 = \frac{2M}{3}(1 - \lambda), \quad \lambda = 1 \\
1 &= 1 + 0 = r_1 + r_2 = \frac{2M}{3}(1 - \lambda), \quad \lambda = \frac{1}{2} \\
1 &= 0 + 1 = r_1 + r_2 = \frac{2M}{3}(1 - \lambda), \quad \lambda = \frac{1}{2} \\
2 &= 2 + 0 = r_1 + r_2 = \frac{2M}{3}(1 - \lambda), \quad \lambda = 0 \\
2 &= 1 + 1 = r_1 + r_2 = \frac{2M}{3}(1 - \lambda), \quad \lambda = 0 \\
2 &= 0 + 2 = r_1 + r_2 = \frac{2M}{3}(1 - \lambda), \quad \lambda = 0 \\
3 &= 3 + 0 = r_1 + r_2 = \frac{2M}{3}(1 - \lambda), \quad \lambda = \frac{-1}{2} \\
3 &= 2 + 1 = r_1 + r_2 = \frac{2M}{3}(1 - \lambda), \quad \lambda = \frac{-1}{2} \\
3 &= 1 + 2 = r_1 + r_2 = \frac{2M}{3}(1 - \lambda), \quad \lambda = \frac{-1}{2} \\
3 &= 0 + 3 = r_1 + r_2 = \frac{2M}{3}(1 - \lambda), \quad \lambda = \frac{-1}{2}
\end{align*}
\]
Example: The row eigenvector matrix $A$ for $M = 3$ balls

With these values for $(r_1, r_2)$, we can assess $f^3_{(r_1, r_2)}(z_1, z_2)$:

$$f^3_{(0, 0)}(z_1, z_2) = (1 + z_1 + z_2)^3 \quad \lambda = 1$$
Example: The row eigenvector matrix $A$ for $M = 3$ balls

With these values for $(r_1, r_2)$, we can assess $f_{(r_1,r_2)}^3(z_1, z_2)$:

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\begin{align*}
    f_{(0,0)}^3(z_1, z_2) &= (1 + z_1 + z_2)^3 \quad \lambda = 1 \\
    f_{(1,0)}^3(z_1, z_2) &= (1 - z_1)^1 (1 + z_1 + z_2)^2 \quad \lambda = \frac{1}{2} \\
    f_{(0,1)}^3(z_1, z_2) &= (1 - z_2)^1 (1 + z_1 + z_2)^2
\end{align*}
\]
Example: The row eigenvector matrix $\mathbf{A}$ for $M = 3$ balls

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\begin{align*}
    f^3_{(0,0)}(z_1, z_2) &= (1 + z_1 + z_2)^3 & \lambda &= 1 \\
    f^3_{(1,0)}(z_1, z_2) &= (1 - z_1)^1(1 + z_1 + z_2)^2 & \lambda &= \frac{1}{2} \\
    f^3_{(0,1)}(z_1, z_2) &= (1 - z_2)^1(1 + z_1 + z_2)^2 \\
    f^3_{(2,0)}(z_1, z_2) &= (1 - z_1)^2(1 + z_1 + z_2)^1 & \lambda &= 0 \\
    f^3_{(1,1)}(z_1, z_2) &= (1 - z_1)^1(1 - z_2)^1(1 + z_1 + z_2)^1 \\
    f^3_{(0,2)}(z_1, z_2) &= (1 - z_2)^2(1 + z_1 + z_2)^1
\end{align*}
\]
Example: The row eigenvector matrix $A$ for $M = 3$ balls

With these values for $(r_1, r_2)$, we can assess $f_{(r_1, r_2)}^3(z_1, z_2)$:

$$
\begin{align*}
\left( \begin{array}{c}
0, 0 \\
1, 0 \\
0, 1 \\
2, 0 \\
1, 1 \\
0, 2 \\
3, 0 \\
2, 1 \\
1, 2 \\
0, 3
\end{array} \right) & \rightarrow \left( \begin{array}{c}
f_0^3(z_1, z_2) = (1 + z_1 + z_2)^3 & \lambda = 1 \\
f_1^3(z_1, z_2) = (1 - z_1)^1 (1 + z_1 + z_2)^2 & \lambda = \frac{1}{2} \\
f_0^3(z_1, z_2) = (1 - z_2)^1 (1 + z_1 + z_2)^2 \\
f_2^3(z_1, z_2) = (1 - z_1)^2 (1 + z_1 + z_2)^1 & \lambda = 0 \\
f_1^3(z_1, z_2) = (1 - z_1)^1 (1 - z_2)^1 (1 + z_1 + z_2)^1 \\
f_0^3(z_1, z_2) = (1 - z_2)^2 (1 + z_1 + z_2)^1 \\
f_3^3(z_1, z_2) = (1 - z_1)^3 & \lambda = -\frac{1}{2} \\
f_2^3(z_1, z_2) = (1 - z_1)^2 (1 - z_2)^1 \\
f_1^3(z_1, z_2) = (1 - z_1)^1 (1 - z_2)^2 \\
f_0^3(z_1, z_2) = (1 - z_2)^3 .
\end{array} \right)
\end{align*}
$$
Example: The row eigenvector matrix $A$ for $M = 3$ balls

Expanding the two polynomials for $\lambda = \frac{1}{2}$ gives us the coefficients and therefore the components of the corresponding eigenvectors:
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$$f^3_{(1,0)}(z_1, z_2) = (1 - z_1)(1 + z_1 + z_2)^2$$

$$= 1 + z_1 + 2z_2 - z_1^2 + 0z_1z_2 + z_2^2 - z_1^3 - 2z_1^2z_2 - z_1z_2^2 + 0z_2^3$$
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and

$$f_{(0,1)}^3(z_1, z_2) = (1 - z_2)(1 + z_1 + z_2)^2$$

$$= 1 + 2z_1 + z_2 + z_1^2 + 0z_1z_2 - z_2^2 + 0z_1^3 - z_1^2z_2 - 2z_1z_2^2 - z_2^3.$$
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$$= 1 + z_1 + 2z_2 - z_1^2 + 0z_1z_2 + z_2^2 - z_1^3 - 2z_1^2z_2 - z_1z_2^2 + 0z_2^3$$

and

$$f_{(0,1)}^3(z_1, z_2) = (1 - z_2)(1 + z_1 + z_2)^2$$
$$= 1 + 2z_1 + z_2 + z_1^2 + 0z_1z_2 - z_2^2 + 0z_1^3 - z_1^2z_2 - 2z_1z_2^2 - z_2^3.$$

The row eigenvectors corresponding to each of these polynomials are

$$(1, 1, 2, -1, 0, 1, -1, -2, -1, 0)$$
Example: The row eigenvector matrix $A$ for $M = 3$ balls

Expanding the two polynomials for $\lambda = \frac{1}{2}$ gives us the coefficients and therefore the components of the corresponding eigenvectors:

\[
f^3_{(1,0)}(z_1,z_2) = (1 - z_1)(1 + z_1 + z_2)^2
\]

\[
= 1 + z_1 + 2z_2 - z_1^2 + 0z_1z_2 + z_2^2 - z_1^3 - 2z_1^2z_2 - z_1z_2^2 + 0z_2^3
\]

and

\[
f^3_{(0,1)}(z_1,z_2) = (1 - z_2)(1 + z_1 + z_2)^2
\]

\[
= 1 + 2z_1 + z_2 + z_1^2 + 0z_1z_2 - z_2^2 + 0z_1^3 - z_1^2z_2 - 2z_1z_2^2 - z_2^3.
\]

The row eigenvectors corresponding to each of these polynomials are

\[
(1, 1, 2, -1, 0, 1, -1, -2, -1, 0)
\]

and

\[
(1, 2, 1, 1, 0, -1, 0, -1, -2, -1).
\]
Example: The row eigenvector matrix $\mathbf{A}$ for $M = 3$ balls

So for $f_3^{(r_1, r_2)}(z_1, z_2)$, we have the row eigenvector matrix

\[
\begin{bmatrix}
1 & z_1 & z_2 & z_1^2 & z_1 z_2 & z_2^2 & z_1^3 & z_1^2 z_2 & z_1 z_2^2 & z_2^3 \\
1 & 3 & 3 & 3 & 6 & 3 & 1 & 3 & 3 & 1 \\
1 & 1 & 2 & -1 & 0 & 1 & -1 & -2 & -1 & 0 \\
1 & 2 & 1 & 1 & 0 & -1 & 0 & -1 & -2 & -1 \\
1 & -1 & 1 & -1 & -2 & 0 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & -1 & -1 & -1 & 0 & 1 & 1 & 0 \\
1 & 1 & -1 & 0 & -2 & -1 & 0 & 0 & 1 & 1 \\
1 & -3 & 0 & 3 & 0 & 0 & -1 & 0 & 0 & 0 \\
1 & -2 & -1 & 1 & 2 & 0 & 0 & -1 & 0 & 0 \\
1 & -1 & -2 & 0 & 2 & 1 & 0 & 0 & -1 & 0 \\
1 & 0 & -3 & 0 & 0 & 3 & 0 & 0 & 0 & -1 \\
\end{bmatrix}
= \mathbf{A}.
\]
Generating function and eigenvalues for $d$ urns

**Theorem**

The coefficients of the functions

$$f_r^M(z) = \left( \prod_{k=1}^{d-1} (1 - z_k)^{r_k} \right) \left( 1 + \sum_{k=1}^{d-1} z_k \right)^{r_d},$$

where all the components of $r$ are nonnegative integers and $\sum_{k=1}^d r_k = M$, define a set of linearly independent eigenvectors of the $M$-ball $d$-urn transition matrix.
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**Corollary**

The eigenvalues of the $d$-urn $M$-ball transition matrix are

$$\lambda = \frac{d}{M(d-1)} r_d - \frac{1}{d-1} \text{ for } r_d \in \{0, \ldots, M\}. \text{ Eigenvalue } \lambda \text{ has multiplicity } \left( \frac{M(d-1)}{d}(1-\lambda)+d-2 \right).$$
Finding the inverse matrix $\Delta^{-1}$ for $d = 3$ urns
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While we cannot expect there to be a pattern in the inverse of the eigenvector matrix, a remarkable pattern emerges upon inspection.
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$d = 3, \ M = 1$:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}^{-1} = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$
Finding the inverse matrix $\Delta^{-1}$ for $d = 3$ urns

While we cannot expect there to be a pattern in the inverse of the eigenvector matrix, a remarkable pattern emerges upon inspection.

$d = 3, M = 1$:

\[
\begin{pmatrix}
1 + z_1 & +z_2 \\
1 - z_1 & +0z_2 \\
1 +0z_1 & -z_2
\end{pmatrix} \quad \frac{1}{3} \quad \begin{pmatrix}
1 + z_1 & +z_2 \\
1 -2z_1 & +z_2 \\
1 +z_1 & -2z_2
\end{pmatrix}
\]
Finding the inverse matrix $\mathbb{A}^{-1}$ for $d = 3$ urns

This pattern suggests that the rows of the inverse matrix $\mathbb{A}^{-1}$ are generated by

$$\tilde{f}^M_{(s_1, s_2)}(z_1, z_2) = \frac{1}{3^M} (1 - 2z_1 + z_2)^{s_1} (1 + z_1 - 2z_2)^{s_2} (1 + z_1 + z_2)^{s_3},$$

where $s_1, s_2, s_3$ are nonnegative integers such that $s_1 + s_2 + s_3 = M$. $s_1, s_2, s_3$ are the same as the $r_1, r_2, r_3$ used to generate the corresponding row of the original eigenvector matrix $\mathbb{A}$. 
Finding the inverse matrix $\mathbb{A}^{-1}$ for $d = 3$ urns

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This function is also a valid solution to our partial differential equation.
The $d$-urn inverse matrix

Theorem

The rows of the matrix $A^{-1}$ are the coefficients of

$$
\tilde{f}_s^M(z) = \frac{1}{d^M} \left( \prod_{k=1}^{d-1} (1 - (d - 1)z_k + \sum_{i \neq k} z_i)^{s_k} \right) \left( 1 + \sum_{k=1}^{d-1} z_k \right)^{M - \sum_{k=1}^{d-1} s_k},
$$

where the row corresponding to the vector $s$ in $\tilde{f}_s^M(z)$ is in the same position as the row corresponding to $r$ in $f_r^M(z)$. 
Proof: $\tilde{f}_s^M(z)$ generates $\mathbb{A}^{-1}$

To prove that $\tilde{f}_s^M(z)$ generates $\mathbb{A}^{-1}$, let the coefficients of $f_r^M(z)$ and $\tilde{f}_s^M(z)$ be $a_{r,s}^{(M)}$ and $\frac{1}{d^M} b_{s,t}^{(M)}$, respectively, so that $\frac{1}{d^M} \mathbb{B}$ is the matrix generated by $\tilde{f}_s^M(z)$. 
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It is sufficient to prove that:

\[
\frac{1}{d^M} \sum_s a_{r,s}^{(M)} b_{s,t}^{(M)} = \begin{cases} 
1 & \text{if } r = t, \\
0 & \text{otherwise}.
\end{cases}
\]

This condition can be proved by induction on \( M \).
Computational complexity

Using diagonalization, we can compute the $n$-step transition matrix $P^n$ more efficiently. Assuming $d \ll M$, 

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Ordinary matrix multiplication: $\sim nM^{3(d-1)}$ operations

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<thead>
<tr>
<th>Balls</th>
<th>100</th>
<th>200</th>
<th>400</th>
</tr>
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<td>0.637</td>
<td>0.786</td>
<td>0.806</td>
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<td>26.861</td>
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(Ordinary matrix multiplication)

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<tbody>
<tr>
<td>10</td>
<td>0.215</td>
<td>0.229</td>
<td>0.234</td>
</tr>
<tr>
<td>20</td>
<td>9.017</td>
<td>9.453</td>
<td>9.655</td>
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(Diagonalization)
Mixing times
Mixing times

The **total variation distance** between a probability distribution $p$ and the stationary distribution $\pi$ is given by

$$d_{TV}(p, \pi) = \frac{1}{2} \sum_i |p_i - \pi_i|.$$
Mixing times

The **total variation distance** between a probability distribution $p$ and the stationary distribution $\pi$ is given by

\[
d_{TV}(p, \pi) = \frac{1}{2} \sum_{i} |p_i - \pi_i|.
\]

The **mixing time** of a Markov chain is the minimum number of steps it takes for the total variation distance to be less than a given $\epsilon$:

\[
t_{mix}(\epsilon) = \min\{n : d_{TV}(P^n, \pi) \leq \epsilon\}
\]
We can bound the mixing time of the Ehrenfest urn model as follows:

\[
\ln \left( \frac{1}{2\epsilon} \right) \left( \frac{1}{1 - \lambda^*} - 1 \right) \leq t_{mix}(\epsilon) \leq -\ln(\epsilon \pi_{min}) \left( \frac{1}{1 - \lambda^*} \right)
\]

where

\[
\lambda^* = 1 - \frac{d}{(d-1)M}
\]

is the second-largest eigenvalue of \( \mathbb{P} \) and

\[
\pi_{min} = \frac{1}{d^M}.
\]
Mixing times: example

Consider $d = 3$ urns, $M = 20$ balls, $\epsilon = 0.25$. If we start with all the balls in one urn, $t_{mix} = 30$. 
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Consider $d = 3$ urns, $M = 20$ balls, $\epsilon = 0.25$. If we start with all the balls in one urn, $t_{mix} = 30$.

The bounds on the mixing time are approximately $8 \leq t_{mix} \leq 312$. 
Mixing times for the multiple-urn Ehrenfest model

Using the eigenanalysis, we can precisely quantify the mixing times for the multiple urn Ehrenfest model.
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Beginning with all balls in an urn, and with the choice of $\epsilon = 0.25$, we have

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<tbody>
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<td>5</td>
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<td>0.2305</td>
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<tr>
<td>10</td>
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When $M = 80$, the dimensions of the transition matrix $\mathbb{P}$ are $\binom{82}{2} \times \binom{82}{2}$, or $3321 \times 3321$. Even equipped with the eigenanalysis, it took R more than 5 hours to compute the total variation distance.
Hitting times

For a Markov chain $X_n$, $n \geq 0$, the hitting time (or first passage time) from state $r$ to state $s$ is the minimum number of steps the chain takes to reach state $s$ for the first time when the chain initially starts at state $r$. The expected value of such a hitting time is denoted by $\mathbb{E}_r [T_s]$, where

$$T_s = \min \{ n \geq 0 : X_n = s, X_i \neq s, \text{for all } i < n \}.$$
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(1) How long does it take to empty a full urn?
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1. How long does it take to empty a full urn?
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We have investigated the following hitting times associated with a multiple urn Ehrenfest model.

(1) How long does it take to empty a full urn?

(2) How long does it take to fill a specific empty urn?

(3) How long does it take to transfer all balls in a full urn to an empty urn?
Computing expected hitting times

General method for computing expected hitting times:
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\[ E_r[T_s] = E[T_s | X_0 = r] \]
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$$\mathbb{E}_r [T_s] = \mathbb{E}[T_s | X_0 = r]$$

$$= 1 + \sum_k \mathbb{E}[T_s | X_1 = k] \times P_{r,k}$$
Computing expected hitting times

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$$\mathbb{E}_r [T_s] = \mathbb{E}[T_s | X_0 = r]$$
$$= 1 + \sum_k \mathbb{E}[T_s | X_1 = k] \times P_{r,k}$$

If we label $\tau_{r,s} = \mathbb{E}_r [T_s]$, it amounts to solve the linear system

$$\tau_{r,s} = 1 + \sum_k \tau_{k,s} \times P_{r,k},$$

and the number of unknowns $\tau_{r,s}$ equals the square of the size of the state space of the Markov chain.
Emptying a full urn, our method
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Given a specific urn containing $M - k$ balls, let $T_k$ be the first time it takes for the urn to have $M - k - 1$ balls.
Emptying a full urn, our method

Given a specific urn containing \( M - k \) balls, let \( T_k \) be the first time it takes for the urn to have \( M - k - 1 \) balls.

\[
T_k = \begin{cases} 
1 & \text{with probability } \frac{M-k}{M}, \\
1 + T'_k & \text{with probability } \frac{k(d-2)}{M(d-1)}, \\
1 + T_{k-1} + T'_k & \text{with probability } \frac{k}{M(d-1)}, 
\end{cases}
\]

where \( T'_k \) is a random variable with the same distribution as \( T_k \).
Emptying a full urn

We find that

$$\mathbb{E}[T_k] = \frac{1}{\binom{M-1}{k}} \sum_{j=0}^{k} \binom{M}{j} \frac{1}{(d-1)^{k-j}}$$
Emptying a full urn

We find that

\[
\mathbb{E}[T_k] = \frac{1}{\binom{M-1}{k}} \sum_{j=0}^{k} \frac{\binom{M}{j}}{(d-1)^{k-j}} = M \int_{0}^{1} x^{M-k-1} \left(\frac{d-x}{d-1}\right)^k \, dx,
\]
Emptying a full urn

We find that

$$\mathbb{E}[T_k] = \frac{1}{(M-1)} \sum_{j=0}^{k} \binom{M}{j} \frac{(d-1)^{k-j}}{(d-1)^{k-j}} = M \int_{0}^{1} x^{M-k-1} \left(\frac{d-x}{d-1}\right)^k \, dx,$$

which gives us

$$\mathbb{E}_{full}[T_{empty}] = \sum_{k=0}^{M-1} \mathbb{E}[T_k]$$
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Filling an empty urn

Redefining $T_k$ as the time to have $k + 1$ balls in an urn initially containing $k$ balls,

$$
\mathbb{E}[T_k] = \frac{d - 1}{(M - 1)} \sum_{j=0}^{k} \binom{M}{j} (d - 1)^{k-j}.
$$
Filling an empty urn

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Using the same method, we find that the expected time to fill a specific empty urn is

$$\mathbb{E}_{empty}[T_{full}] = M(d - 1) \sum_{k=0}^{M-1} \frac{d^k}{k + 1}.$$
Filling an empty urn

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E[T_k] = \frac{d - 1}{M - 1} \sum_{j=0}^{k} \binom{M}{j}(d - 1)^{k-j}.
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Using the same method, we find that the expected time to fill a specific empty urn is

\[
E_{\text{empty}}[T_{\text{full}}] = M(d - 1) \sum_{k=0}^{M-1} \frac{d^k}{k + 1}.
\]

Starting with a full urn, the time to fill any of the other urns is

\[
E_{\text{full}}[T_{\text{full\_any}}] = M \sum_{k=0}^{M-1} \frac{d^k}{k + 1}.
\]
Hitting time graphical representation

**blue**: time to empty a specific urn
**orange**: time to fill a specific urn
Summary of results
Summary of results

(1) Developed a generating function $f_r^M$ for the eigenvectors of the transition matrix $P$, first for the 3-urn model, then generalized for $d$ urns. $f_r^M$ also allows us to solve for the eigenvalues $\lambda$. 
Summary of results

(1) Developed a generating function $f_r^M$ for the eigenvectors of the transition matrix $\mathbb{P}$, first for the 3-urn model, then generalized for $d$ urns. $f_r^M$ also allows us to solve for the eigenvalues $\lambda$.

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(4) Solved for general formulae for hitting times for various scenarios of the Ehrenfest urn model.
Works cited


