k-cleaning of Dessin d'Enfants

Gabrielle Melamed, Jonathan Pham, Austin Wei

Willamette University Mathematics Consortium REU

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Outline

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Definition

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Some Riemann Surfaces

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Let $f : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ be defined by $f(z) = z^n$.

$f'(z) = nz^{n-1}$, which is 0 at 0 and ∞ at ∞, and we can check that $f(0) = 0$ and $f(\infty) = \infty$. So f is in fact a Belyi map.

Belyi's Theorem

Theorem (Belyi, 1979)

A Riemann surface X can be defined over $\overline{\mathbb{Q}}$ if and only if X admits a Belyi map.

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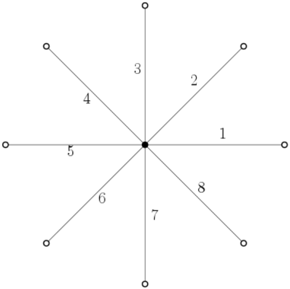
The algebraic numbers feature in a mathematical object of some importance known as the absolute Galois group. One major application of Belyi's theorem is in tying the structure of Belyi maps to the structure of the absolute Galois group.

Dessins from Belyi maps

Given a Belyi map $f : X \to \mathbb{P}^1(\mathbb{C})$, we can define an embedded bicolored graph on X:

$$f^{-1}(0) := \text{the set of black vertices}$$
$$f^{-1}(1) := \text{the set of white vertices}$$
$$f^{-1}(\infty) := \text{the interiors of faces}$$
$$f^{-1}((0,1)) := \text{the set of edges}$$



A Two-face Dessin

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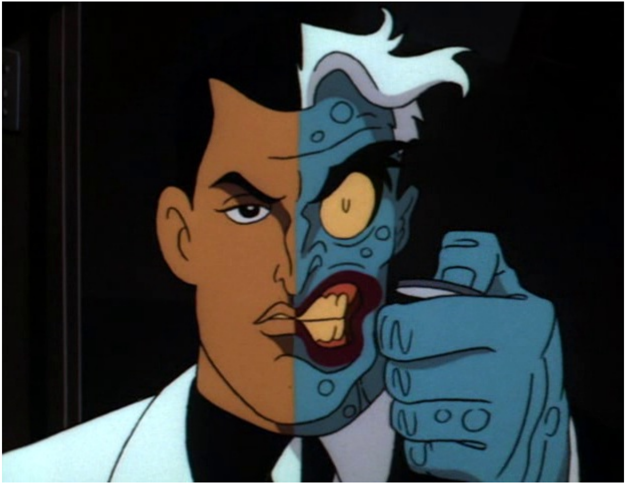


A Two-faced Dessin

Define $g(z) = \frac{-(z-1)^2}{4z}$. g has critical points at $z = -1, 0, 1$, for critical values of $1, \infty, 0$ respectively, so g is in fact a Belyi map. Its corresponding dessin has two faces.

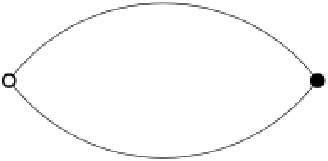


Figure: The dessin corresponding to g

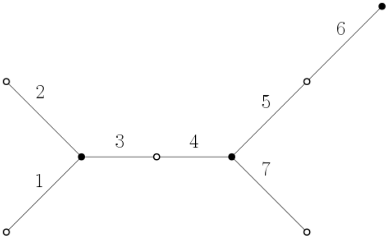
Outline

Dessins

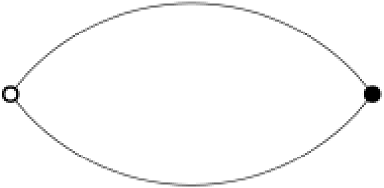
Definition

A **dessin d'enfant** (henceforth "dessin") is a bicolored graph embedded into a Riemann surface, whose orientation induces an ordering of edges around the vertices.

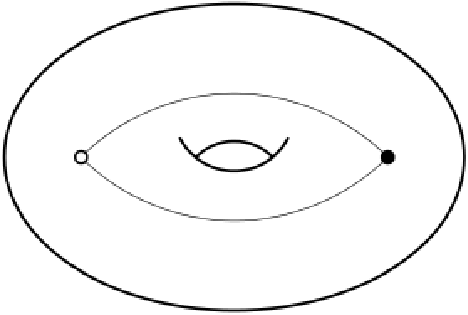
Example



A Two-faced Dessin

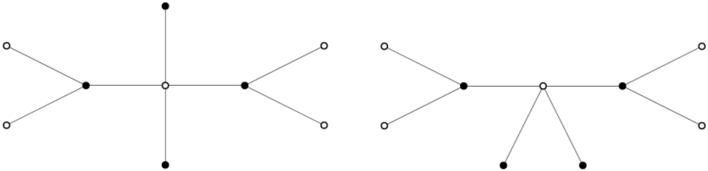


But not...



Example

How the dessin is embedded into the surface also distinguishes between dessins.



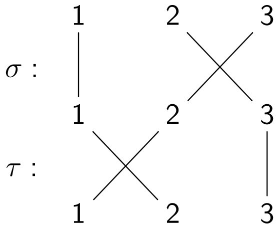
Permutation Groups: The Symmetric Group

Definition

The **Symmetric Group** of degree n, denoted S_n, is the set of all bijections from $\{1, 2, \ldots, n\}$ to itself, with the binary operation given by composition. Elements of S_n are called **permutations**. The size, or the **order** of S_n is denoted $|S_n| = n!$.

Example

Define:



We can write σ and τ in cycle-notation as: $\sigma = (1)(2,3)$ and $\tau = (1,2)(3)$.
Their product is $\sigma\tau = \tau \circ \sigma = (1,2,3)$ in cycle notation.

Even and Odd Permutations

Permutations can be even or odd, like the integers. One visualization is that a permutation is odd if the number of crossings in its diagram is odd (and likewise for an even permutation). For example, σ and τ from the previous slide are odd, and their product $(1, 2, 3)$ is even.

Even and odd permutations behave like even and odd numbers: namely, the composition of two even or two odd permutations is an even permutation, and the composition of one odd and one even permutation is an odd permutation.

Permutation Groups

Definition

A **Permutation Group** is a subgroup of S_n: it is a nonempty subset $H \subseteq S_n$ which is closed under products and inverses. That is, for $\sigma, \tau \in H$, $\sigma\tau \in H$ and $\sigma^{-1}, \tau^{-1} \in H$.

Example

The subset $H = \{\mathrm{id}, (1,2,3), (1,3,2)\}$ defines a subgroup of S_3, where $\mathrm{id} = (1)(2)(3)$.

The Alternating Group

The set of all even permutations inside S_n also forms a subgroup! We call this the alternating group, and denote it by A_n.

For example, A_4 consists of the permutations: $(1,2,3)$, $(1,3,2)$, $(1,2,4)$, $(1,4,2)$, $(1,3,4)$, $(1,4,3)$, $(2,3,4)$, $(2,4,3)$, $(1,2)(3,4)$, $(1,3)(2,4)$, $(1,4)(2,3)$, and id.

Permutation Groups: Generating Sets

An arbitrary subset of S_n need not be a subgroup. For example, the set $\{(1, 2, 3)\}$ is closed under neither inverses nor products.

What if we add in all of its powers? Then we get the subset $H = \{\mathrm{id}, (1, 2, 3), (1, 3, 2)\}$ from the previous slide, which is a subgroup of S_3.

In general, given a subset $X \subseteq S_n$, we denote by $\langle X \rangle$ the subgroup "generated by X", which is the smallest subgroup of S_n that contains X.

For example, $H = \langle (1, 2, 3) \rangle$.

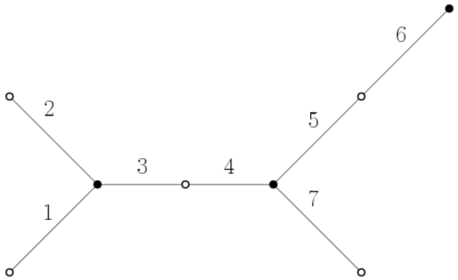
Edgy Permutations

The labeling of the edges of the dessin allows us to define permutations in S_n corresponding to a dessin:

$$\begin{aligned} \sigma_B := &\ \text{the permutation which rotates edges counterclockwise} \\ &\ \text{about the black vertices} \\ \sigma_W := &\ \text{the permutation which rotates edges counterclockwise} \\ &\ \text{about the white vertices} \end{aligned}$$

Example

For the following dessin,



$$\sigma_B = (1,3,2)\,(4,7,5)$$
$$\sigma_W = (3,4)\,(5,6)$$

Monodromy Group

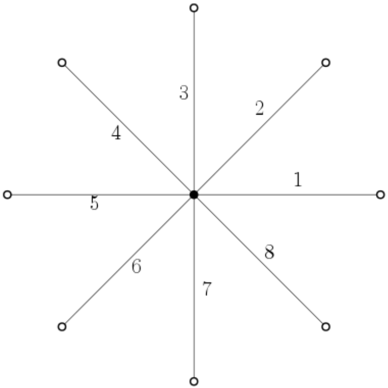
Definition

The **monodromy group** of a dessin is $G := \langle \sigma_B, \sigma_W \rangle$, where σ_B and σ_W are the two permutations corresponding to the dessin.

The monodromy group is a subgroup of the symmetric group, and for connected dessins, this subgroup is transitive: for any numbers $x, y \in \{1, 2, \ldots, n\}$, there is some permutation $\sigma \in G$ such that $\sigma(x) = y$.

We sometimes denote the monodromy group of a dessin D by $\mathrm{Mon}\,(D)$.

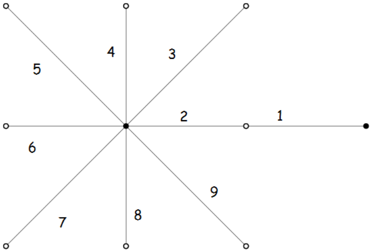
Example



For this dessin, $\sigma_B = (1, 2, 3, 4, 5, 6, 7, 8)$, $\sigma_W = \mathrm{id}$, and thus $G = \langle(1, 2, 3, 4, 5, 6, 7, 8)\rangle \cong \mathbb{Z}_8$, the cyclic group of order 8.

A musing on the monodromy group

Determining the monodromy group of a given dessin is not so easy in general. For example, suppose we added an edge to the 8-star from before.



Now $\sigma_B = (2,3,4,5,6,7,8,9)$ and $\sigma_W = (1,2)$ so that $G = S_9$

Motivation behind composition

Cayley's theorem tells us that *every group is isomorphic to a subgroup of some symmetric group*. So each of our monodromy groups sits inside some symmetric group S_n (where n is the number of edges in the dessin).
However, the order of S_n grows pretty fast (like n! fast!), so it becomes difficult to determine these subgroups.

Motivation behind composition

Is there a way to decompose a dessin into smaller, more familiar dessins in order to work with nicer monodromy groups which sit inside smaller symmetric groups?

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Of course there is!
How does this decomposition reflect in the monodromy group of the original dessin?

Composition

We can construct trees with a composition process given by Adrianov and Zvonkin:

- Start with two trees, P and Q, pictured below:

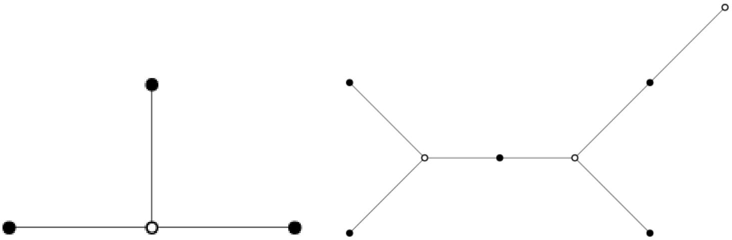

Figure: P on the left, Q on the right.

Composition

We can construct trees with a composition process given by Adrianov and Zvonkin:

- We begin the composition $P \circ Q$ by first distinguishing two vertices of P: label them a square and a triangle.

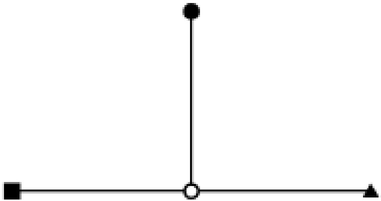

Figure: P: pick two vertices to be the square and triangle.

Composition

We can construct trees with a composition process given by Adrianov and Zvonkin:

- Mark every black vertex of Q with a square and similarly every white vertex of Q with a triangle.

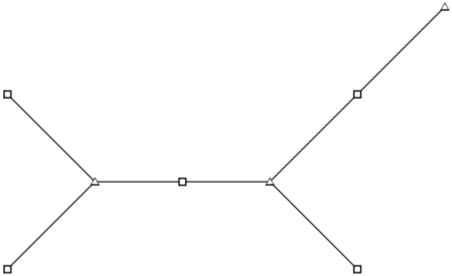

Figure: Q: change black vertices to squares and whites to triangles.

Composition

We can construct trees with a composition process given by Adrianov and Zvonkin:

- Finally, replace every edge of Q with the tree P, matching the square vertex of P to the square vertex of that edge, and likewise for the triangles.

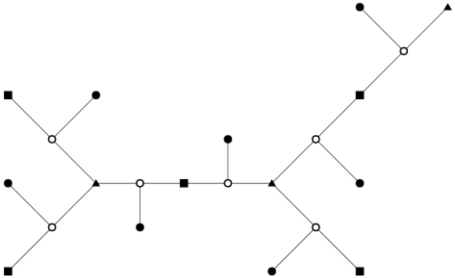

Figure: Q: change black vertices to squares and whites to triangles.

Example

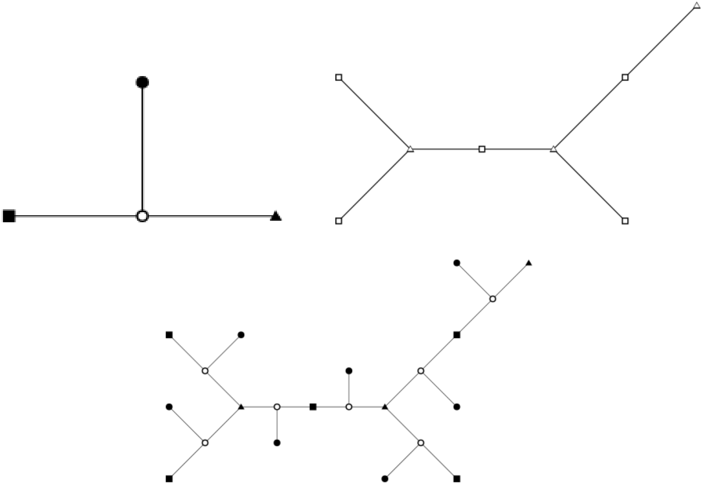

Figure: P, Q, and $P \circ Q$ respectively

Composition

This process is significant to us because
$\mathrm{Mon}\,(P \circ Q) \leq \mathrm{Mon}\,(Q) \wr \mathrm{Mon}\,(P)$ [A. Zvonkin].

This is a consequence of the Krasner-Kaloujnine Embedding Theorem, which states that if a group G is an extension of H by N, it is isomorphic to a subgroup of the wreath product $H \wr N$.

Wreath Products: a Formal Definition

So, what's a wreath product?

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Definition

Let G and H be groups, let n be a positive integer and define a homomorphism $\varphi : G \to S_n$. Have K be the direct product of n copies of H.

If $\psi : S_n \to \mathrm{Aut}(K)$ is an injective homomorphism which lets the elements of S_n permute the n factors of K and $\phi = \psi \circ \varphi$ is a homomorphism from G into $\mathrm{Aut}(K)$ we say that the **wreath product** of H by G is the semi-direct product $K \rtimes G$ with respect to ϕ and is denoted $H \wr G$.

Wreath Products

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Consider $S_4 \wr S_3$ as a representation of an order of three group presentations, each of which has four members (with the assumption that the members each speak just once).

The first thing in the wreath product corresponds to group members: within each group, the members are free to speak in any order.

The second thing corresponds to the number of groups, ordering the groups into those presenting first, second, and third.

Example

Consider the wreath product $S_3 \wr \mathbb{Z}_2$.

What do its elements look like? A pair of permutations, along with an element of $\mathbb{Z}_2$. For example, for $\sigma, \tau \in S_3$, both $(\sigma, \tau, 0)$ and $(\sigma, \tau, 1)$ are in $S_3 \wr \mathbb{Z}_2$.

How do we multiply them? The multiplication is component-by-component, but if the last component of the second is a 1, we switch the components of the first before multiplying. For example:

$$(\sigma, \tau, 0) \cdot (\sigma, \tau, 1) = (\tau\sigma, \sigma\tau, 1) \text{ while}$$
$$(\sigma, \tau, 1) \cdot (\sigma, \tau, 0) = (\sigma^2, \tau^2, 1).$$

Wreath Products: A Remark about Embedding

Given a wreath product $S_n \wr \mathbb{Z}_m$, the wreath product will embed naturally into a symmetric group of degree mn.

For example, $S_3 \wr \mathbb{Z}_2$ embeds as $\langle(1,2),(1,2,3),(4,5),(4,5,6),(1,4)(2,5)(3,6)\rangle$ (note that $S_3 = \langle(1,2),(1,2,3)\rangle$).

That is, we partition the set $\{1,\ldots,mn\}$ into m sets of size n, and envision one copy of S_n acting on each set. The final ingredient is the wreath element $(\mathrm{id},\ldots,\mathrm{id},1)$ which cycles between the copies.

Cleaning

One particular composition which is fairly well studied in the literature is known as cleaning, which is composition with a dessin of the following form:

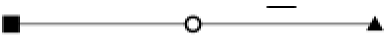

Figure: The 2-star which we plug into the edges of our dessin. For some mysterious reason, it is labeled with “blank” and “bar”.

Given a dessin, its cleaned form is a dessin where every original vertex is colored black, and a white vertex is inserted on every edge.

Some Clean Dessins

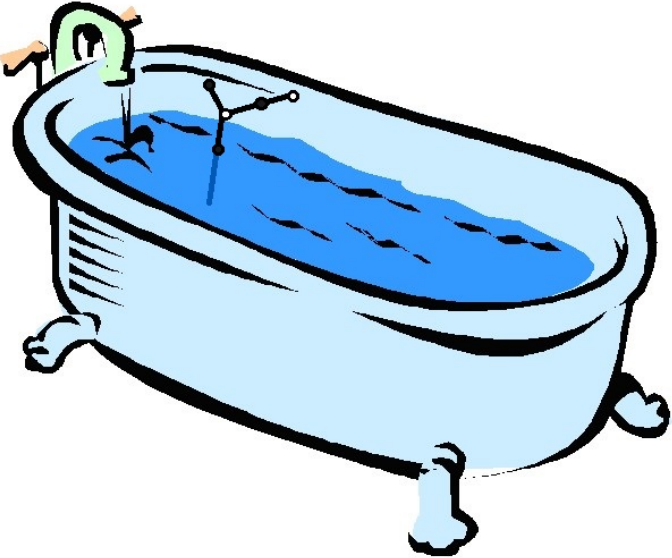

Figure: A dessin freshening up before its date.

Some Clean Dessins

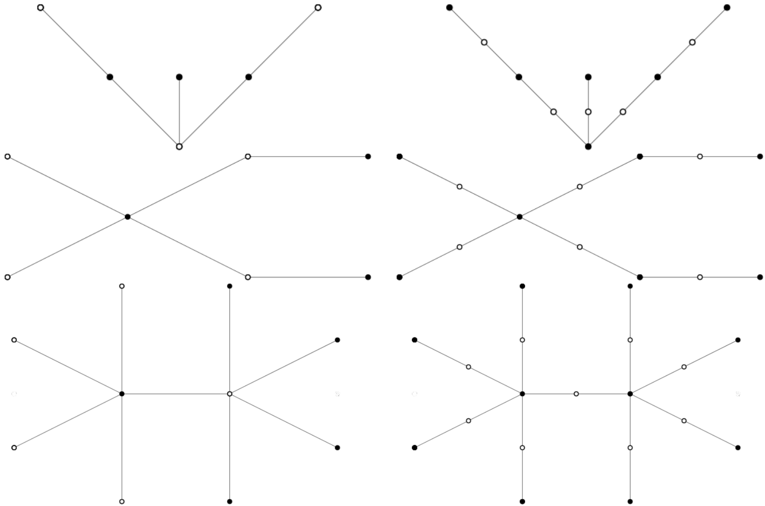

Figure: Here are some dessins before / after cleaning.

Outline

Cleaning is Nice!

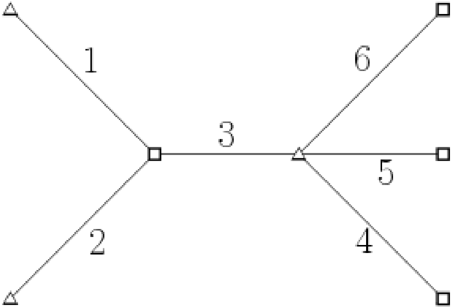

Figure: A pre-cleaning dessin.

$$\sigma_B = (1, 2, 3)$$
$$\sigma_W = (3, 4, 5, 6)$$

This dessin has monodromy group S_6!

Cleaned Dessin

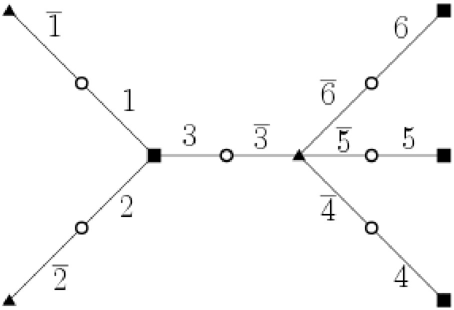

Figure: The cleaned dessin.

$$\sigma_B = (1, 2, 3)(\overline{3}, \overline{4}, \overline{5}, \overline{6})$$
$$\sigma_W = (1, \overline{1})(2, \overline{2}) \cdots (6, \overline{6})$$

This dessin has monodromy group $S_6 \wr \mathbb{Z}_2$.

The Clean Embedding in General

In general, given a dessin D with permutations π_B and π_W around the black and white vertices respectively, the generators of the cleaned dessin embed into the wreath product $\mathrm{Mon}\,(D) \wr \mathbb{Z}_2$ as follows:

$$\begin{aligned} \sigma_B &\longmapsto (\pi_B, \pi_W, 0) \\ \sigma_W &\longmapsto (\mathrm{id}, \mathrm{id}, 1) \end{aligned}$$

k-cleaning

One natural question is whether we can compose with a tree larger than the 2-star but still preserve the nice embedding, where σ_B has cycles only between "blanks" or "bars" and σ_W cycles between all the different forms of one number.
We can do this via a composition we call k-cleaning!

k-cleaning

Definition

A dessin (tree) is k-**cleaned** if it is the result of a composition with a k-star, where the square and the triangle are both children of the central vertex.

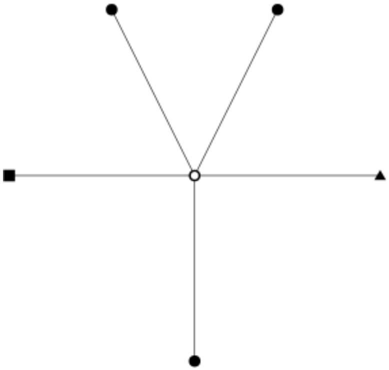

Figure: One possible 5-star

k-cleaning

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- σ_W still cycles through the alter egos of a single number: because a white vertex corresponds to exactly one edge of the original dessin, and because composition preserves ordering, σ_W will be a product of k-cycles, one for every edge.

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- σ_W still cycles through the alter egos of a single number: because a white vertex corresponds to exactly one edge of the original dessin, and because composition preserves ordering, σ_W will be a product of k-cycles, one for every edge.
- σ_B is still the product of the two original cycles, each between only one type (e.g. “blanks” or “bars”)

k-cleaning Embedding

In general, a k-star can have $0 \leq j \leq k-2$ edges below the path from square to triangle.
The embedding for k-cleaning with a k-star that has j edges below the path from square to triangle is:

$$\sigma_B \longmapsto (\sigma_1, \mathrm{id}, \ldots, \mathrm{id}, \sigma_2, \mathrm{id}, \ldots, \mathrm{id}, 0)$$
$$\sigma_W \longmapsto (\mathrm{id}, \ldots, \mathrm{id}, 1)$$

where σ_2 is in the $(j+2)$th component.

Applications of k-cleaning

Algebra Lemma

Lemma

Suppose that $\sigma_0, \sigma_1 \in S_n$ *with* $\langle \sigma_0, \sigma_1 \rangle \geq A_n$ *with* $n \geq 5$. *Let* $k \geq 2$, *and define* $x_1 = (\sigma_0, \sigma_1, \mathrm{id}, \dots, \mathrm{id}), \dots, x_{k-1} = (\mathrm{id}, \dots, \mathrm{id}, \sigma_0, \sigma_1), x_k = (\sigma_1, \mathrm{id}, \dots, \mathrm{id}, \sigma_0) \in S_n^k$, *and* $G = \langle x_1, \dots, x_k \rangle$. *If* $k > 2$, G *must contain* A_n^k. *If* $k = 2$, *the same claim holds as long as* $|\sigma_0| \neq |\sigma_1|$.

Proof sketch.

We show that under the given assumptions, there exist elements of the form $\rho = (\mathrm{id}, \dots, \mathrm{id}, \rho_i, \mathrm{id}, \dots, \mathrm{id})$ where $\rho_i \neq \mathrm{id}$. For example, if $k = 2$ and $r = |\sigma_0| \neq |\sigma_1| = t$, then $x_1^t = (\sigma_0^r, \mathrm{id}, \dots, \mathrm{id})$.
We then look at all conjugates of this element (permutation-tuples which are of the form $\tau^{-1}\rho\tau$) and show that G must contain a conjugacy class, and hence must be normal.
It must thus contain at least $\mathrm{id} \times \mathrm{id} \times A_n \times \mathrm{id} \times \cdots \times \mathrm{id}$ for each component, from which it follows that $A_n^k \leq G$. □

A 2-cleaned dessin with unique degree sequence

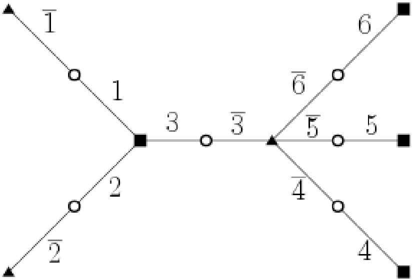

Figure: Dessin D with degree sequence: $[r, t, 1^{r+t-2}; 2^{r+t-1}]$; here, $r = 3$, $t = 4$.

The embedding gives that:

$$\sigma_B \longmapsto ((1,2,3),(3,4,5,6),0)$$
$$\sigma_W \longmapsto (\mathrm{id}, \mathrm{id}, 1)$$

2-cleaned dessin (cont.)

Taking $\sigma_0 = (1,2,3)$ and $\sigma_1 = (3,4,5,6)$, we see that $x_1 = \sigma_B$, $x_2 = \sigma_W^{-1}\sigma_B\sigma_W$, and the lemma immediately gives that $A_6 \wr \mathbb{Z}_2 \leq \text{Mon}\,(D)$.

A quick check via the embedding verifies that both the first and second components independently contain odd permutations, which lets us conclude that $\text{Mon}\,(D) \cong S_6 \wr \mathbb{Z}_2$.

A 3-cleaned dessin with unique degree sequence

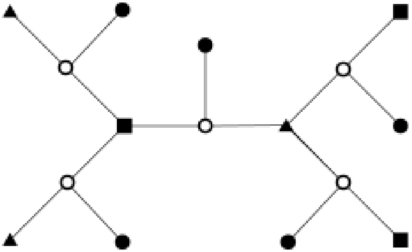

Figure: Dessin D with degree sequence: $[r^2, 1^{4r-3}; 3^{2r-1}]$; here, $r = 3$.

The embedding gives that:

$$\sigma_B \longmapsto ((1,2,3),(3,4,5),\mathrm{id},0)$$
$$\sigma_W \longmapsto (\mathrm{id},\mathrm{id},\mathrm{id},1)$$

3-cleaned dessin (cont.)

Again, applying the lemma with $\sigma_0 = (1, 2, 3), \sigma_1 = (3, 4, 5)$ and $k = 3$ immediately gives $A_6 \wr \mathbb{Z}_3 \leq \mathrm{Mon}\,(D)$.

In this case, as both σ_0 and σ_1 are even, the other containment gives $A_6 \wr \mathbb{Z}_3 \cong \mathrm{Mon}\,(D)$.

Outline

- A check through the list of degree sequences corresponding to exactly two trees given by Adrianov (2009) reveals that six of twelve correspond to k-cleaned trees. The lemma can probably be used to compute monodromy groups for many of these cases.

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- A check through the list of degree sequences corresponding to exactly two trees given by Adrianov (2009) reveals that six of twelve correspond to k-cleaned trees. The lemma can probably be used to compute monodromy groups for many of these cases.
- Extending the lemma to other simple groups (other than A_n), as well as for non-simple groups, which the proof suggests should have monodromy groups which are normal.
- The monodromy group of a composition is not in general the full wreath product of the monodromy groups of its factors. k-cleaning gives some examples of both monodromy groups which are proper subgroups of the wreath product and ones which are the full wreath product, and may help determine some conditions under which the monodromy group is a proper subgroup.

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