2'. [Extra problem: $C$ is the unit circle, $\oint_C y\,dx - x\,dy$.] Parametrize the circle by $r(t) = \langle \cos t, \sin t \rangle$ on $[0, 2\pi]$. We get $\oint_C y\,dx - x\,dy = \int_0^{2\pi} [(\sin t)(-\sin t) - (\cos t)(\cos t)]\,dt = \int_0^{2\pi} -1\,dt = -2\pi$.

$Q_x = -1$ and $P_y = 1$, so by Green’s Theorem, we have $\int_D (-1)\,dA = \int_0^{2\pi} \int_0^1 (-2)r\,dr\,d\theta = \int_0^{2\pi} \int_0^1 (21^2 - 0^2)\,d\theta = \int_0^{2\pi} -1\,d\theta = -2\pi$. I’ll be darned! It’s the same!

$\int_1^1 \int_{3\pi}^3 (4y^3 - 2x^2y)\,dy\,dx = \int_0^1 y^4 - x^2y^2\Big|_{y=3x}^{y=x} = \int_0^1 [(81-9x^2)-(81x^4-9x^4)]\,dx = \int_0^1 81 - 9x^2 - 72x^4 \,dx = 81x - 3x^3 - \frac{318}{5}$.

7. The parabolas meet at $x = 0$ and $x = 1$; on this interval, $y$ ranges from $x^2$ to $\sqrt{x}$. $Q_x = 2$ and $P_y = 1$, so we get $\int_0^1 \int_{x^2}^{\sqrt{x}} (2 - 1)\,dy\,dx = \int_0^1 \sqrt{x} - x^2\,dx = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$.

8. This looks like a job for Polar Coordinates! $Q_x = 4x^3 + 4xy^2$ and $P_y = 0$, so we get $\int_0^{2\pi} \int_1^2 [4r^3 \cos^3 \theta + 4r^3 \cos \theta \sin^2 \theta]r\,dr\,d\theta = \int_0^{2\pi} \int_0^1 4r^4 \cos \theta \sin \theta \,dr\,d\theta = \int_0^{2\pi} \cos \theta \,d\theta \int_0^1 4r^4 \,dr = 0$ since integrating $\cos \theta$ over its period gives 0.

9. More polar coordinates! $Q_x = -3x^2, P_y = 3y^2$, so we get $\int_0^{2\pi} \int_0^2 -3(x^2 + y^2)\,r\,dr\,d\theta = \int_0^{2\pi} d\theta \int_0^2 -3r^3\,dr = 2\pi \left( -\frac{3}{4} \cdot \frac{1}{2^4} \right) = -24\pi$.

11. The given orientation of $C$ is negative, so we will need a minus sign. $Q_x = 2x$ and $P_y = 3y^2$, so we get $-\int_0^\pi \int_0^1 (2x - 3y^2)\,dy\,dx = -\int_0^\pi 2x \sin x - \sin^3 x\,dx = -\int_0^\pi 2x \sin x - \sin x(1 - \cos^2 x)\,dx$. We can integrate the first term by parts ($u = x, dv = \sin x$) and the rest directly or by substitution. We get $-2(-x \cos x + \sin x) + \cos x - \frac{1}{3} \cos^3 x$. Evaluation from 0 to $\pi$ gives $-2\pi + 1 + \frac{1}{3} = \frac{4}{3} - 2\pi$.

12. This is also oriented negatively, so we will need a minus sign. $x$ ranges from 0 to 2 and $y$ from 0 to 3. $Q_x = 2x + 2y \cos x$ and $P_y = 2y \cos x$, so we get $-\int_0^2 \int_0^3 2xy\,dy\,dx = -\int_0^2 6x^2\,dx = -16$.

13. Clockwise gives a negative orientation. $Q_x = -y^2, P_y = x^2$. We get $-\int_0^{2\pi} \int_0^5 (-y^2 - x^2)\,r\,dr\,d\theta = \int_0^{2\pi} d\theta \int_0^5 r^3\,dr = 2\pi \frac{625}{4} = \frac{625\pi}{2}$.

14. Finally, a positive orientation! $Q_x = \frac{2}{1 + (y/x)^2}(-y/x^2) = \frac{-2y}{x^2 + y^2}$ and $P_y = 1 - \frac{2y}{x^2 + y^2}$. We get $\int_D -1\,dA$. This just gives the negative of the area of the circle, which is $\pi$, so the integral is $-\pi$. How lucky is that?

Just for practice, here is an alternative: Let $u = x - 2, v = y - 3$. The curve becomes $u^2 + v^2 = 1$, and $\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$. The double integral is now $\int \int (1-1)[1]\,du\,dv = \int_0^{2\pi} \int_0^1 -1\,r\,dr\,d\theta = -2\pi(1/2) = -\pi$, where $U$ is the disk of radius 1 centered at the origin.

17. The given curve is oriented positively. We have $Q_x = y^2$ and $P_y = x$. The region is bounded by the $x$- and $y$-axes and the line $y = 1 - x$. The work done is $W = \oint_C F \cdot dr = \int_D (y^2 - x)\,dA =$
\[ \int_0^1 \int_0^{1-x} (y^2 - x) \, dy \, dx = \int_0^1 \frac{1}{3} (1-x)^3 - x(1-x) \, dx = -\frac{1}{12} (0-1) - \frac{1}{2} + \frac{1}{3} = -\frac{1}{12}. \]

19. The width of an arch of the cycloid is $2\pi$. As $t$ increases from 0 to $2\pi$, we travel along the top of the cycloid. To close the path, we then must travel left along the $x$-axis from $2\pi$ to 0. This is a negative orientation, so we will need a minus sign; I will use the second formula. This has $dx = x'(t) \, dt = 1 - \cos t$, so we get
\[
\int_0^{2\pi} (1 - \cos t)(1 - \cos t) \, dt = \int_0^{2\pi} (1 - 2 \cos t + \cos^2 t) \, dt = \int_0^{2\pi} 1 + \frac{1 + \cos 2t}{2} \, dt = (2\pi + \pi) = 3\pi.
\]

The second formula has a minus sign, but we also had a negative orientation, so the two cancelled. Also notice that I dropped the $\cos t$ and $\cos 2t$ from the integral; both of these integrate to 0 on $[0, 2\pi]$. 