Solutions to Homework Assignment 31

MATH 249-01 and -02
Section 16.5, Page 1096
1, 3, 5, 7, 9, 10, 11, 12, 13, 15, 17

1. (a) \( \nabla \times \mathbf{F} = < -x^2, 3xy, -xz > \).
(b) \( \nabla \cdot \mathbf{F} = yz + 0 + 0 = yz \).

2. (a) \( \nabla \times \mathbf{F} = < x - y, -y, 1 > \).
(b) \( \nabla \cdot \mathbf{F} = 0 + z - \frac{1}{2\sqrt{z}} = z - \frac{1}{2\sqrt{z}} \). Note that the answer in the back of the book is wrong.

3. (a) \( \nabla \times \mathbf{F} = < 0, 0, 0 > \).
(b) \( \nabla \cdot \mathbf{F} = e^x \sin y - e^y \sin y + 1 = 1 \).

4. (a) \( \nabla \times \mathbf{F} = < 1/y, -1/x, 1/x > \).
(b) \( \nabla \cdot \mathbf{F} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \).

For 9, 10, and 11, I will assume \( \mathbf{F} \) has the form \( \mathbf{F} = < P, Q, 0 > \).

5. Notice that \( \mathbf{F} \) does not change as we move left to right, so \( P = 0 \) and \( F = < 0, Q, 0 > \). However, \( \mathbf{F} \) gets shorter and shorter as \( z \) increases, so \( Q_y < 0 \). Thus \( \nabla \cdot \mathbf{F} = Q_y < 0 \). For the \( \nabla \times \mathbf{F} \), we need only consider the \( z \)-component, which is \( Q_z - P_y \). (This is because \( \mathbf{F} \) is independent of \( z \).) Since \( \mathbf{F} \) and \( \mathbf{Q} \) are independent of \( x \), \( Q_z = 0 \). Also, as we move up, the \( x \)-components do not change. Therefore, \( Q_z - P_y = 0 \), so \( \nabla \times \mathbf{F} = 0 \).

6. Here, \( P_x > 0 \) and \( Q_y > 0 \) since the vectors get longer the farther we get from the origin. Thus \( \nabla \cdot \mathbf{F} = P_x + Q_y > 0 \). Notice that as we move to the right, the \( y \)-components of the vectors do not change. This means that \( Q_x = 0 \). Also, as we move up, the \( x \)-components do not change. Therefore, \( Q_x - P_y = 0 \), so \( \nabla \times \mathbf{F} = 0 \).

7. As we move from left to right, the \( x \)-components of the vectors remain constant, so \( P_x = 0 \). As we move up, the \( x \)-components increase, so \( P_y > 0 \). The \( y \)-components are all zero, so \( Q = 0 \). Now \( \nabla \cdot \mathbf{F} = 0 \) and \( \nabla \times \mathbf{F} = < 0, 0, Q_x - P_y > = < 0, 0, -P_y > \). Since \( P_y > 0 \), this points down.

8. (a) This is meaningless; \( \nabla \cdot \mathbf{F} \) only applies to vector fields, not scalar functions.
(b) This is a vector field; its components are the first partial derivatives of \( f \).
(c) This is a scalar field.
(d) This is a vector field.
(e) This is meaningless; the gradient only applies to scalar functions.
(f) This is a vector field.
(g) This is a scalar field.
(h) This is meaningless; \( \nabla \times \mathbf{F} \) is meaningless; \( \nabla \cdot \mathbf{F} \) is a scalar field, so we can’t cross it with anything.
(i) This is a vector field.
(j) This is meaningless; \( \nabla \cdot \mathbf{F} \) only applies to vector fields and \( \nabla \times \mathbf{F} \) is a scalar field.
(k) This is meaningless; \( \nabla \cdot \mathbf{F} \) is a scalar field, so we can’t cross it with anything.
(l) This is a scalar field.

9. \( \nabla \times \mathbf{F} = < x - x, -(y - y), z - z > = < 0, 0, 0 > \). Therefore, the vector field is conservative. If \( \nabla f = \mathbf{F} \), then \( f_x = yz \), so \( f = xyz + g(y, z) \). Also, \( f_y = xz \) implies \( f = xyz + h(x, z) \) and \( f_z = xy \) implies \( f = xyz + k(x, y) \). We may take \( f(x, y, z) = xyz \).

10. \( \nabla \cdot \mathbf{F} = < 2y - 2y, -(0 - 0), 2x - 2x > = < 0, 0, 0 > \). Therefore, \( \mathbf{F} \) is conservative. Now \( f_x = 2xy \), so \( f = x^2y + g(y, z) \). With \( f_y = x^2 + 2yz \), we get \( f = x^2y + y^2z + h(x, z) \), and with \( f_z = y^2 \), we get \( f = y^2z + k(x, y) \). Comparing these, we see that \( f(x, y, z) = x^2y + y^2z \) will work.

11. \( \nabla \times \mathbf{F} = < 0 - 0, -(0 - 0), -e^{-x} - e^{-x} > = < 0, 0, -2e^{-x} > \), so \( \mathbf{F} \) is not conservative.