1. (a) Since \( \frac{dy}{dt} = -y + 5 \), we have

\[
\frac{dy}{dt} = -y + 5 \quad \frac{d}{dt} \left[ \ln | -y + 5 | \right] = 1 \\
\ln | -y + 5 | = t + C \\
| -y + 5 | = e^{t+C} \\
| -y + 5 | = e^{-t} \\
y + 5 = \pm C e^{-t}.
\]

Notice that I have used the same symbol \( C \) throughout to represent the constant, even though the \( C \) at the end is actually \( -e^{-C} \). (Check it!) I will continue with this practice.

Since \( y(0) = 5 \pm C = y_0 \), we must have \( C = \pm (y_0 - 5) \). Thus, \( y = 5 + (y_0 - 5)e^{-t} \).

(b) Here we have \( \frac{dy}{dt} = -2y + 5 \), so

\[
\frac{dy}{dt} = -2y + 5 \quad \frac{-1}{2} \frac{d}{dt} \left[ \ln | -2y + 5 | \right] = 1 \\
\frac{d}{dt} \left[ \ln | -2y + 5 | \right] = -2 \\
| -2y + 5 | = C e^{-2t} \\
-2y + 5 = \pm Ce^{-2t} \\
y = \frac{5}{2} \pm Ce^{-2t}.
\]

With \( y(0) = y_0 \), we have \( \frac{5}{2} \pm C = y_0 \), so \( C = \pm (y_0 - 5/2) \). Thus, \( y = \frac{5}{2} \pm \left(y_0 - \frac{5}{2}\right) e^{-2t} \).

(c) \( \frac{dy}{dt} = -2y + 10 \), so

\[
\frac{dy}{dt} = -2y + 10 \quad \frac{-1}{2} \frac{d}{dt} \left[ \ln | -2y + 10 | \right] = 1 \\
\frac{d}{dt} \left[ \ln | -2y + 10 | \right] = -2 \\
| -2y + 10 | = C e^{-2t} \\
-2y + 10 = \pm Ce^{-2t} \\
y = 5 \pm Ce^{-2t}.
\]

With \( y(0) = y_0 \), we have \( 5 \pm C = y_0 \), so \( C = \pm (y_0 - 5) \). Thus, \( y = 5 + (y_0 - 5)e^{-2t} \).

Note that in all cases the long-term behavior of all solutions is to approach the equilibrium solution, which is 5 in parts (a) and (c), and 5/2 in part (b). Also, because \( e^{-2t} \) decreases faster than \( e^{-t} \), the solutions in (b) and (c) approach equilibrium more quickly than that in (a).
2. Compare the results of this exercise with Exercise 1.

(a) Since \( \frac{dy}{dt} = y - 5 \), we have

\[
\frac{dy}{dt} = y - 5 = 1
\]

\[
\frac{d}{dt} [\ln |y - 5|] = 1
\]

\[
\ln |y - 5| = t + C
\]

\[
|y - 5| = e^{t+C}
\]

\[
y - 5 = \pm Ce^t
\]

\[
y = 5 \pm Ce^t.
\]

Since \( y(0) = 5 \pm C = y_0 \), we must have \( C = \pm(y_0 - 5) \). Thus, \( y = 5 + (y_0 - 5)e^t \).

(b) Here we have \( \frac{dy}{dt} = 2y - 5 \), so

\[
\frac{dy}{dt} = 2y - 5 = 1
\]

\[
\frac{1}{2} \frac{d}{dt} [\ln |2y - 5|] = 1
\]

\[
\frac{d}{dt} [\ln |2y - 5|] = 2
\]

\[
|2y - 5| = Ce^{2t}
\]

\[
2y - 5 = \pm Ce^{2t}
\]

\[
y = \frac{5}{2} \pm Ce^{2t}.
\]

With \( y(0) = y_0 \), we have \( \frac{5}{2} \pm C = y_0 \), so \( C = \pm(y_0 - 5/2) \). Thus, \( y = \frac{5}{2} \pm \left( y_0 - \frac{5}{2} \right) e^{2t} \).

(c) \( \frac{dy}{dt} = 2y - 10 \), so

\[
\frac{dy}{dt} = 2y - 10 = 1
\]

\[
\frac{1}{2} \frac{d}{dt} [\ln |2y - 10|] = 1
\]

\[
\frac{d}{dt} [\ln |2y - 10|] = 2
\]

\[
|2y - 10| = Ce^{2t}
\]

\[
2y - 10 = \pm Ce^{2t}
\]

\[
y = 5 \pm Ce^{2t}.
\]

With \( y(0) = y_0 \), we have \( 5 \pm C = y_0 \), so \( C = \pm(y_0 - 5) \). Thus, \( y = 5 + (y_0 - 5)e^{2t} \).

Note that in all cases the long-term behavior of all solutions except the equilibrium solution is to diverge from the equilibrium solution, which is 5 in parts (a) and (c), and 5/2 in part (b). Also, because \( e^{2t} \) increases faster than \( e^t \), the solutions in (b) and (c) diverge from equilibrium more quickly than that in (a).
6. (a) From Example 1, we have \( p = 900 - 50e^{t/2} \). Extinction occurs when \( p = 0 \), so we solve:
\[
900 = 50e^{t/2}
\]
\[
18 = e^{t/2}
\]
\[
t/2 = \ln 18
\]
\[
t = 2\ln 18
\]
\[t \approx 5.78.
\]
The population will be extinct in about 5.78 months.

(b) We now have \( p(0) = p_0 = 900 + c \), so \( c = p_0 - 900 \). Thus \( p = 900 + (p_0 - 900)e^{t/2} \). Setting this equal to zero and solving yields
\[
(900 - p_0)e^{t/2} = 900
\]
\[
e^{t/2} = \frac{900}{900 - p_0}
\]
\[
t/2 = \ln \left( \frac{900}{900 - p_0} \right)
\]
\[t = 2\ln \left( \frac{900}{900 - p_0} \right).
\]

(c) For the population to be extinct in 1 year (12 months), we need
\[
0 = 900 + (p_0 - 900)e^{12/2}
\]
\[
p_0 = 900(e^6 - 1)e^{-6}
\]
\[
p_0 \approx 898.
\]

If the population begins with 898 mice, it will be extinct in about one year.

11. First, let’s solve the differential equation \( Q' = -rQ \). We have \( \frac{Q'}{Q} = -r \), so \( \ln |Q| = -rt + C \). Since \( Q \) must be positive, we may drop the absolute value bars. Now \( Q = Ce^{-rt} \). We don’t know what \( C \) is, but we do know that \( Q(0) = C \) and therefore \( Q(\tau) = \frac{C}{2} \). (This is what it means to say that \( \tau \) is the half-life.) Thus \( Ce^{-r\tau} = \frac{C}{2} \); notice that the \( C \)'s cancel out – the initial amount is irrelevant!

We now have \( e^{-r\tau} = \frac{1}{2} \), so \(-r\tau - \ln 2\), and \( r\tau = \ln 2 \), as desired.

12. If the amount is reduced by one-quarter, it is reduced to three-quarters. The form of a solution to the equation \( Q' = -rQ \) is given in Exercise 11, above, as \( Q = Ce^{-rt} \). We are told that the half-life is 1620 years, and we know that 1620\(r = \ln 2 \). Thus, we need to solve:
\[
Ce^{-t(ln2)/1620} = \frac{3}{4}C
\]
\[
\frac{t\ln 2}{1620} = \ln \frac{3}{4}
\]
\[t = \frac{\ln(4/3)}{\ln 2} \cdot 1620
\]
\[t \approx 672.36.
\]
The amount of radium-226 is reduced by 1/4 in about 672 years.

13. (a) We solve as follows:
\[
RQ' + \frac{Q}{C} = V
\]
\[
RQ' = V - \frac{Q}{C}
\]
\[
\frac{RQ'}{V - Q/C} = 1
\]
\[-CR \frac{d}{dt} \ln |V - Q/C| = 1\]
\[\frac{d}{dt} \ln |V - Q/C| = -\frac{1}{CR} \]
\[
\ln |V - Q/C| = -\frac{1}{CR} t + c
\]
\[
V - Q/C = \pm ce^{-t/(CR)}
\]
\[
Q = CV \pm ce^{-t/(CR)}.
\]

Whew!

Since \(Q(0) = 0\), we have \(CV \pm c = 0\), so \(c = \pm CV\). Therefore \(Q(t) = CV - CV e^{-t/(CR)}\). The graph is below; I set \(C = V = R = 1\) so MAPLE could plot it.

(b) Since \(\lim_{t \to \infty} Q(t) = CV\), \(Q_L = CV\).

(c) Now we are giving an “initial” value of \(Q_L = CV\) at \(t = t_1\). That is, we have \(Q(t) = \pm ce^{-t/(CR)}\) for \(t \geq t_1\). At \(t = t_1\), this is \(CV = \pm ce^{-t_1/(CR)}\), so \(c = \pm CV e^{t_1/(CR)}\). This gives \(Q(t) = CV e^{t_1/(CR)} e^{-t/(CR)} = CV e^{(t-t_1)/(CR)}\). I set \(t_1 = 1\) for MAPLE’s sake.

15. In part (c), the units are grams, so we will use grams throughout.

(a) We are told that \(q(0) = 5000\) g, and the pool contains 60000 gallons of water. The concentration of dye at time \(t\) is \(\frac{q(t)}{60000}\) g/gal, and the dye is removed from 200 gallons per minute. Thus at time \(t\), \(\frac{q}{60000}(200) = \frac{q}{300}\) g of dye are being removed per minute. That is, \(q'(t) = -\frac{q(t)}{300}\), and \(q(0) = 5000\).

(b) We have

\[
\frac{q'(t)}{q(t)} = -\frac{1}{300}
\]
\[
\frac{d}{dt} \ln |q(t)| = -\frac{1}{300}
\]
\[
\ln |q(t)| = -\frac{t}{300} + C
\]
\[
q(t) = Ce^{-t/300}.
\]
Notice that we have dropped the absolute value bars since \( q \) must be positive. With \( q(0) = C = 5000 \), we have \( q(t) = 5000e^{-t/300} \).

(c) Since 4 hours is 240 minutes, the amount of dye will be \( 5000e^{-240/300} \approx 2246.64 \) g, making the concentration \( 2246.64/60000 \approx 0.037 \) g/gal. Prepare for some green guests.

(d) We need to have \( 0.02 = \frac{5000e^{-t/300}}{60000} = \frac{e^{-t/300}}{12} \). Solving for \( t \) gives about 428 minutes, or a little over 7 hours. Bummer!

(e) Let \( a \) represent the flow rate. Then \( q(t) = 5000e^{-at/60000} \). (See part (a) for the effect of the flow rate; the 300 in the denominator of the exponent came from reducing 200/60000.) With \( t = 240 \), we need \( e^{-240a/60000} = 0.02 \). Solving gives \( a \approx 356.78 \) gal/min. (The back of the book has 256.78; I believe this is a typo.)