1. \( e^{1+2i} = e(\cos 2 + i \sin 2) \)
2. \( e^{2-3i} = e^2(\cos(-3) + i \sin(-3)) = e^2(\cos 3 - i \sin 3) \).
3. \( e^{3\pi} = \cos \pi + i \sin \pi = -1 \).
4. \( e^{2-i\pi/2} = e^2(\cos(\pi/2) - i \sin(\pi/2)) = -ie^2 \).
5. \( 2^1-i = e^{(i-1)ln2} = e^{-ln2}(\cos(ln2) + i \sin(ln2)) = \frac{1}{2} (\cos(ln2) + i \sin(ln2)) \).
6. \( e^{-1+2i} = e^{(-1+2i)ln\pi} = e^{-ln\pi}(\cos(2ln\pi) + i \sin(2ln\pi)) = \frac{1}{\pi} (\cos(2ln\pi) + i \sin(2ln\pi)) \).

7. The characteristic equation is \( r^2 - 2r + 2 = 0 \), so \( r = \frac{2 \pm \sqrt{-4}}{2} = 1 \pm i \). Thus, the general solution is \( e^t(c_1 \cos t + c_2 \sin t) \).
8. We have \( r^2 + 2r - 8 = 0 \), so \( r = -4 \) or \( r = 2 \). Thus, \( y(t) = c_1 e^{-4t} + c_2 e^{2t} \).
9. We have \( r^2 + 6r + 13 = 0 \), so \( r = \frac{-6 \pm \sqrt{-16}}{2} = -3 \pm 2i \). Thus, \( y(t) = e^{-3t}(c_1 \cos 2t + c_2 \sin 2t) \).
10. We have \( r^2 + 2r + 1.25 = 0 \), so \( r = \frac{-2 \pm \sqrt{-1}}{2} = -1 \pm \frac{i}{2} \). Thus, \( y(t) = e^{-t}(c_1 \cos(t/2) + c_2 \sin(t/2)) \).
11. We have \( r^2 + r + 1.25 = 0 \), so \( r = -\frac{1}{2} \pm i \). Thus, \( y(t) = e^{-t/2}(c_1 \cos t + c_2 \sin t) \).
12. We have \( r^2 + 4 = 0 \), so \( r = \pm 2i \). Thus \( y(t) = c_1 \cos 2t + c_2 \sin 2t \). With \( y(0) = 0 \), we get \( c_1 = 0 \). With \( y'(0) = 1 \), we get \( 2c_2 \cos 2t = 1 \), so \( c_2 = 1/2 \). Therefore, \( y(t) = \frac{1}{2} \sin 2t \). Its graph is below; the function just continues to oscillate at the same amplitude as \( t \to \infty \).
13. We have \( r^2 - 2r + 5 = 0 \), so \( r = \frac{2 \pm \sqrt{-16}}{2} = 1 \pm 2i \). Thus \( y(t) = e^t(c_1 \cos 2t + c_2 \sin 2t) \). With \( y(\pi/2) = 0 \), we have \( c_1 = 0 \), so \( y(t) = c_2 e^t \sin 2t \). This implies \( y'(t) = c_2 e^t \sin 2t + 2c_2 e^t \cos 2t \). With \( y'(\pi/2) = 2 \), we get \( -2c_2 e^{t/2} = 2 \), so \( c_2 = -e^{-t/2} \). Therefore, \( y(t) = -e^{-t/2} \sin 2t \). This is an exponentially growing oscillation.
14. We have \( r^2 + r + 1.25 = 0 \), so \( r = -\frac{1}{2} \pm \frac{\sqrt{-4}}{2} = -\frac{1}{2} \pm i \). The general solution is therefore \( y(t) = e^{-t/2}(c_1 \cos t + c_2 \sin t) \). With \( y(0) = 3 \), we have \( c_1 = 3 \), so \( y(t) = e^{-t/2}(3 \cos t + c_2 \sin t) \). Thus \( y'(t) = -\frac{1}{2} e^{-t/2}(3 \cos t + c_2 \sin t) + e^{-t/2}(-3 \sin t + c_2 \cos t) \). Since \( y'(0) = 1 \), we get \( -\frac{3}{2} + c_2 = 0 \), so \( c_2 = \frac{3}{2} \). Therefore, \( y(t) = e^{-t/2} \left( \cos t - \frac{3}{2} \sin t \right) \). This is an exponentially decaying oscillation.
24. (a) We have \(5r^2 + 2r + 7 = 0\), so \(r = \frac{-2 \pm \sqrt{-36}}{10} = \frac{-1 \pm i\sqrt{34}}{5}\). Thus \(u(t) = e^{-t/5}(c_1 \cos(\sqrt{34}t/5) + c_2 \sin(\sqrt{34}t/5))\). \(u(0) = c_1 = 2\), so \(u(t) = e^{-t/5}(2 \cos(\sqrt{34}t/5) + c_2 \sin(\sqrt{34}t/5))\). \(u'(t) = -\frac{1}{5}e^{-t/5}((2 \cos(\sqrt{34}t/5) + \frac{\sqrt{68}}{5} \sin(\sqrt{34}t/5) + \frac{\sqrt{34}c_2}{5} \cos(\sqrt{34}t/5))\). \(u'(0) = -2/5 + \frac{7}{\sqrt{34}} = 1\), so \(c_2 = 7/\sqrt{34}\). Thus \(u(t) = e^{-t/5}\left(2 \cos(\sqrt{34}t/5) + \frac{7}{\sqrt{34}} \sin(\sqrt{34}t/5)\right)\).

(b) According to MAPLE, \(|u(t)| < 0.1\) for all \(t > 14.511\) (approximately). (I graphed and zoomed in until I could read off the \(t\)-coordinate fairly precisely.)

28. (a) We have already seen both that \(\cos t\) and \(\sin t\) are solutions of \(y'' + y = 0\) and that their Wronskian is nonzero.

(b) If \(y = e^t\), then \(y' = ie^t\) and \(y'' = -e^t\). Thus \(y'' + y = -e^t + e^t = 0\), so \(y = e^t\) is a solution.

Since \(c_1 \cos t + c_2 \sin t\) is the general solution of \(y'' + y = 0\), we must have \(e^t = c_1 \cos t + c_2 \sin t\).

(c) Now \(e^0 = 1 = c_1\).

(d) \(ie^t = -\sin t + c_2 \cos t\). With \(t = 0\), we get \(i = c_2\).

29. \(\frac{e^t + e^{-t}}{2} = \frac{\cos t + i \sin t + \cos t - i \sin t}{2} = \cos t\). \(\frac{e^t - e^{-t}}{2} = \frac{\cos t + i \sin t - \cos t + i \sin t}{2} = \sin t\).

33. Let \(a\) and \(b\), \(a < b\), be consecutive zeroes of \(y_1\), and assume that \(y_2\) is never zero between \(a\) and \(b\). Then on the interval \((a, b)\), \(y_2\) is either strictly positive or strictly negative; without loss of generality, we may assume that \(y_2\) is positive on \((a, b)\).

Consider now \(y(t) = \frac{y_1(t)}{y_2(t)}\). Since \(y_2(t)\) is nonzero on \((a, b)\), this function is defined and differentiable on \((a, b)\). Notice that \(y'(t) = \frac{y_1'(t)y_2(t) - y_1(t)y_2'(t)}{[y_2(t)]^2}\), and the Wronskian of \(y_2\) and \(y_1\) appears in the numerator. Since \(y_1\) and \(y_2\) are linearly independent, this is nonzero on \((a, b)\). However, \(y(a) = 0\) and \(y(b) = 0\), so by Rolle’s theorem, \(y'(c) = 0\) for some \(c \in (a, b)\).

This means that \(y_2\) must be zero somewhere in \((a, b)\).

We may argue similarly that between consecutive zeroes of \(y_2\) there must be a zero of \(y_1\), so there is one and only one zero of \(y_2\) between consecutive zeroes of \(y_1\).

35. First, \(\frac{q' + 2pq}{2q^{3/2}} = \frac{-2te^{-t^2} + 2te^{-t^2}}{2e^{-3t^2/2}} = 0\), which is constant. Let \(x = inte^{-t^2/2}dt\). Then we have \(e^{-t^2} \frac{d^2y}{dx^2} + (1 - e^{-t^2} + te^{-t^2/2}) \frac{dy}{dx} + e^{-t^2} y = 0\). This simplifies to \(\frac{d^2y}{dx^2} + y = 0\), which has solution \(y = c_1 \cos x + c_2 \sin x\), where \(x = \int e^{-t^2/2}dt\).

36. First, \(\frac{q' + 2pq}{2q^{3/2}} = \frac{3 + 6t^3}{2t^3}\), which is not constant. We will not be able to use this technique on this DE.

37. Rewrite: \(y'' + \frac{t^2 - 1}{t} y' + t^2 y = 0\). Then \(\frac{q' + 2pq}{2q^{3/2}} = \frac{2t + 2(t)(t^2 - 1)}{2t^3} = 1\), so this will work.

Let \(x = \int t dt = \frac{1}{2} t^2\). Then \(t^2 \frac{d^2y}{dt^2} + (1 + (t^2 - 1)) \frac{dy}{dx} + t^2 y = 0\) gives \(\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0\), so we get the characteristic equation \(r^2 + r + 1 = 0\). Our roots are \(r = \frac{-1 \pm i\sqrt{3}}{2}\). The solution is therefore \(e^{-t^2/2}(c_1 \cos(\sqrt{3}x/2) + c_2 \sin(\sqrt{3}x/2)) = e^{-t^2/4}(c_1 \cos(\sqrt{3}t^2/4) + c_2 \sin(\sqrt{3}t^2/4))\).