1. The general solution is \( y = c_1 \cos x + c_2 \sin x \). Since \( y(0) = 0 \), we require \( c_1 = 0 \). Since \( y'(\pi) = 1 \), we require \( c_2(-1) = 1 \), so \( c_2 = -1 \). We therefore have the unique solution \( y(x) = -\sin x \).

2. The general solution is \( y = c_1 \cos \sqrt{2}x + c_2 \sin \sqrt{2}x \). \( y' = -\sqrt{2}c_1 \sin \sqrt{2}x + \sqrt{2}c_2 \cos \sqrt{2}x \). Since \( y'(0) = 1 \), we require \( \sqrt{2}c_2 = 1 \); thus, \( c_2 = \frac{1}{\sqrt{2}} \). Since \( y'(\pi) = 0 \), we require \( -\sqrt{2}c_1 \sin \sqrt{2}\pi + \cos \sqrt{2}\pi = 0 \); therefore, \( c_1 = \frac{1}{\sqrt{2}} \cos \sqrt{2}\pi \). The general solution is \( y(x) = \frac{1}{\sqrt{2}} \cot \sqrt{2}\pi \cos \sqrt{2}x + \frac{1}{\sqrt{2}} \sin \sqrt{2}x \).

3. The general solution is \( y(x) = c_1 \cos x + c_2 \sin x \). Since \( y(0) = 0 \), we have \( c_1 = 0 \). Since \( y(L) = 0 \), we have \( c_2 \sin L = 0 \). This has no nontrivial solution if \( L \neq n\pi \) for some integer \( n \), and it has the nontrivial solution \( c_2 \sin x \) if \( L = n\pi \) for some integer \( n \).

4. This has the same general solution as Number 3. Since \( y'(0) = 1, c_2 = 1 \). Thus \( y(x) = c_1 \cos x + \sin x \). \( y(L) = c_1 \cos L + \sin L = 0 \), so \( c_1 = -\tan L \) unless \( \cos L = 0 \). If \( \cos L = 0 \), then \( \sin L \neq 0 \), so there is no solution. In the first case, the solution is \( y(x) = -\tan L \cos x + \sin x \).

5. We have \((D^2 + 1)D^2y = 0\), which has general solution \( c_1 + c_2x + c_3 \cos x + c_4 \sin x \). The sine and cosine are solutions of the corresponding homogeneous equation, so we just have \( y_p(x) = c_1 + c_2x \). This gives \( y''_p = 0 \), so \( 0 + c_1 + c_2x = x \). Therefore, \( c_1 = 0 \) and \( c_2 = 1 \).

We have \( y(x) = x + c_3 \cos x + c_4 \sin x \). Since \( y(0) = 0 \), this means \( c_3 = 0 \). The condition \( y(\pi) = 0 \) then implies that \( \pi = 0 \), so there is no solution. (But we got a nice review!)

6. We have \((D^2 + 2)D^2y = 0\), so the solution has the form \( c_1 + c_2x + c_3 \cos x + c_4 \sin x \). The solutions of the corresponding homogeneous equation are \( \sin \sqrt{2}x \) and \( \cos \sqrt{2}x \), so \( y_p = c_1 + c_2x \) and \( y''_p = 0 \). We get \( 2(c_1 + c_2x) = x \), so \( c_1 = 0 \) and \( c_2 = 1/2 \). Now \( y(x) = \frac{1}{2} x + c_3 \cos \sqrt{2}x + c_4 \sin \sqrt{2}x \).

\( y(0) = 0 \), so \( c_3 = 0 \). \( y(\pi) = 0 = \frac{\pi}{2} + c_4 \sin(\sqrt{2}\pi) \). Thus \( c_4 = -\frac{\pi}{2 \sin \sqrt{2}\pi} \). Therefore, \( y(x) = \frac{1}{2} x - \frac{\pi}{2 \sin \sqrt{2}\pi} \sin(\sqrt{2}x) \).

7. We have \((D^2 + 4)(D^2 + 1)y = 0\), giving a solution of the form \( c_1 \cos 2x + c_2 \sin 2x + A \cos x + B \sin x \). The \( \cos 2x \) and \( \sin 2x \) terms solve the corresponding homogeneous equation, so \( y_p = A \cos x + B \sin x \) and \( y''_p = -A \cos x - B \sin x \). Therefore,

\[
\cos x = y''_p + 4y_p = 3A \cos x + 3B \sin x.
\]

This gives \( A = \frac{1}{3} \) and \( B = 0 \), so \( y(x) = \frac{1}{3} \cos x + c_1 \cos 2x + c_2 \sin 2x \).

Now \( y(0) = 0 \), so \( \frac{1}{3} + c_1 = 0 \), and \( c_1 = -\frac{1}{3} \). \( y(\pi) = 0 \), as well, so \( -\frac{1}{3} + c_1 = 0 \), and \( c_1 = \frac{1}{3} \). Uh-oh! This has no solution.

8. We may reuse most of our work from the last problem; the only difference is that \( A = 0 \) and \( B = \frac{1}{3} \).

Thus \( y(x) = \frac{1}{3} \cos x + c_1 \cos 2x + c_2 \sin 2x \).

Now \( y(0) = 0 \), so \( c_1 = 0 \), and \( c_1 = \frac{1}{3} \). \( y(\pi) = 0 \), as well, so \( -\frac{1}{3} + c_1 = 0 \), and \( c_1 = \frac{1}{3} \). Uh-oh! This has no solution.

9. We may reuse our work from Number 7: \( y(x) = \frac{1}{3} \cos x + c_1 \cos 2x + c_2 \sin 2x \).

Since \( y'(0) = 0 \), we have \( c_2 = 0 \). Since \( y'(\pi) = 0 \), we have again that \( c_2 = 0 \). Therefore, the solution is \( y(x) = \frac{1}{3} \cos x + c_1 \cos 2x \), and again there are infinitely many solutions.
10. We may again borrow from earlier work; we just replace the 2’s in the prior problems with $\sqrt{3}$’s and the $\frac{1}{3}$ with a $\frac{1}{2}$: 
$$y(x) = \frac{1}{2} \cos x + c_1 \cos \sqrt{3}x + c_2 \sin \sqrt{3}x.$$ 
Now $y'(0) = 0$ implies that $c_2 = 0$. $y'(\pi) = 0$ implies that $-c_1 \sqrt{3} \sin \sqrt{3}\pi = 0$, so $c_1 = 0$ as well. Thus $y(x) = \frac{1}{2} \cos x$.

11. Assume first that $\lambda > 0$. The general solution is $y(x) = c_1 \cos \sqrt{\lambda}x + c_2 \sin \sqrt{\lambda}x$. With $y(0) = 0$, we have $c_1 = 0$. With $y'(\pi) = 0$, we have $\sqrt{\lambda} c_2 \cos(\sqrt{\lambda}\pi) = 0$. In order to get a nontrivial solution, we need to have $\sqrt{\lambda}\pi = \frac{(2n + 1)\pi}{2}$ for some integer $n$. Thus, the eigenvalues are $\frac{(2n + 1)^2}{4}$ for $n$ a nonnegative integer. The corresponding eigenfunctions are $y_n(x) = c_2 \sin \frac{2n + 1}{2}x$.

If $\lambda = 0$, we have $y'' = 0$, so $y = c_1 + c_2x$. This cannot satisfy the boundary conditions, so $\lambda = 0$ is not an eigenvalue.

If $\lambda < 0$, we have $y(x) = c_1 e^{\sqrt{\lambda}x} + c_2 e^{-\sqrt{\lambda}x}$. It is not hard to see that the only solution of this form is the trivial solution.

12. For $\lambda > 0$, the general solution is $y(x) = c_1 \cos \sqrt{\lambda}x + c_2 \sin \sqrt{\lambda}x$. With $y'(0) = 0$, we have $c_2 = 0$. With $y(\pi) = 0$, we have $c_1 \cos \sqrt{\lambda}\pi = 0$. To get a nontrivial solution, we must have $\sqrt{\lambda}\pi = \frac{(2n + 1)\pi}{2}$ for some integer $n$. Thus, the eigenvalues are $\frac{(2n + 1)^2}{4}$ for $n$ a nonnegative integer. The corresponding eigenfunctions are $y_n(x) = c_1 \cos \frac{2n + 1}{2}x$.

Again, there are no negative eigenvalues and zero is not an eigenvalue.

13. For $\lambda > 0$, the general solution is $y(x) = c_1 \cos \sqrt{\lambda}x + c_2 \sin \sqrt{\lambda}x$. With $y'(0) = 0$, we have $c_2 = 0$. With $y'(\pi) = 0$, we have $-c_1 \sqrt{\lambda} \sin \sqrt{\lambda}\pi = 0$. To get a nontrivial solution, we must have $\sqrt{\lambda}\pi = n\pi$ for some integer $n$. Thus, the eigenvalues are $n^2$ for $n$ a positive integer. The corresponding eigenfunctions are $y_n(x) = c_1 \cos nx$.

If $\lambda = 0$, then $y(x) = c_1 + c_2x$, so $y'(0) = 0$ implies that $c_2 = 0$. Now $y(x) = c_1$ will also satisfy $y'(\pi) = 0$, so $y(x) = c_1$ is an eigenfunction for the eigenvalue 0.