1. (a) Closure fails: \(1 + 3 = 4\), but \(4 \notin S\).
(b) Only 0 has an additive inverse.
5. I will apply the two-step subring test in each case.

(a) Let \(A = \begin{bmatrix} 0 & r \\ 0 & 0 \end{bmatrix}\), \(B = \begin{bmatrix} 0 & s \\ 0 & 0 \end{bmatrix}\) \(\in M(\mathbb{R})\) with \(r, s \in \mathbb{Q}\). Then \(A - B = \begin{bmatrix} 0 & r - s \\ 0 & 0 \end{bmatrix}\) is also such a matrix since \(r - s \in \mathbb{Q}\) and \(AB = \begin{bmatrix} 0 & r \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & s \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\) is, too, since \(0 \in \mathbb{Q}\). Thus, this is a subring. There is no identity since every product gives the zero matrix.

(b) Certainly if \(A\) and \(B\) are two such matrices, so is \(A - B\) since the 2,1-entry remains a zero and all other entries of the sum remain integers. If \(A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\) and \(B = \begin{bmatrix} d & e \\ 0 & f \end{bmatrix}\), then \(AB = \begin{bmatrix} ad & ae + bf \\ 0 & cf \end{bmatrix}\), which is another such matrix. Therefore this is a subring of \(M(\mathbb{R})\). It has the identity \(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\).

(c) A quick check confirms that if \(A\) and \(B\) are two such matrices, then so are \(A - B\) and \(AB\). There is no identity: if \(A = \begin{bmatrix} a & a \\ b & b \end{bmatrix}\), then \(AB = \begin{bmatrix} a(r + s) & a(t + u) \\ b(r + s) & b(t + u) \end{bmatrix}\). Thus we need \(r + s = 1\) and \(t + u = 1\). On the other hand, \(\begin{bmatrix} r & s \\ t & u \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\) is another such matrix. Therefore this is a subring of \(M(\mathbb{R})\). It has the identity \(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\).

(d) This is a subring with unity (from \(a = 1\)).
(e) This is also a subring with unity (use \(a = 1\) again).

9. Let \(z_1 = a + b\sqrt{2}, z_2 = c + d\sqrt{2} \in \mathbb{Z}[\sqrt{2}]\). Then \(z_1 - z_2 = (a - c) + (b - d)\sqrt{2} \in \mathbb{Z}[\sqrt{2}]\) and \(z_1z_2 = (ac + 2bd) + (ad + bd)\sqrt{2} \in \mathbb{Z}[\sqrt{2}]\), so \(\mathbb{Z}[\sqrt{2}]\) is a ring by the two-step subring test.

16. Notice that \(R = \{3k|k \in \mathbb{Z}_{18}\}. If a, b \in R, then a = 3k, b = 3l for some a, b \in \mathbb{Z}_{18}\). Now \(a - b = 3(k - l) \in R\) and \(ab = 3(3kl) \in R\), so \(R\) is a ring by the two-step subring test. It does not have an identity.

24. To complete the multiplication table, we rely on the distributive law: \(s \cdot t = s(s + s) = s \cdot s + s \cdot s = t + t + t = s\).
Similarly, \(t \cdot s = (s + s) \cdot s = s\) and \(t \cdot t = (s + s) \cdot (s + s) = s \cdot s + s \cdot s + s \cdot s + s \cdot s = t + t + t + t = s + s + s = t\).

The multiplication table is thus

\[
\begin{array}{ccc}
  r & s & t \\
  r & r & r \\
  s & r & t \\
  t & r & s \\
\end{array}
\]

25. \(xy = x(x + x) = xx + xx = y + y = w. xz = x(x + y) = xx + xy = y + w = y. yx = (x + x)x = xx + xx = y + y = w. yz = y(x + y) = yx + yy = w + w = w. zx = (y + x)x = yx + xx = w + y = y.\)

The table is

\[
\begin{array}{cccc}
  w & x & y & z \\
  w & w & w & w \\
  x & w & y & y \\
  y & w & w & w \\
  z & w & y & w \\
\end{array}
\]
29. (a) $R \times S$ is not an integral domain: if $R = \mathbb{Z}_2$ and $S = \mathbb{Z}_2$, then $(0,1) \cdot (1,0) = (0,0)$ even though neither $(0,1)$ nor $(1,0)$ is zero in $R \times S$.

   (b) This is also false. Using the same example as in (a), we see that $(1,1)$ is the multiplicative identity. However, $(1,0) \cdot (x,y) = (x,0) \neq (1,1)$ for any $y$, so $(1,0)$ has no multiplicative inverse.

30. Let $x \in \mathbb{R}$. If $x \leq 2$, then $f(x)g(x) = 0(2-x) = 0$. If $x > 2$, then $f(x)g(x) = (x-2) \cdot 0 = 0$. Thus $f(x)g(x) = 0$ for all $x \in \mathbb{R}$, so $fg = 0_R$.

34. (a) Since $b \neq 0$, $bx = 1_R$ has a solution. Now $bb = b \implies (bb)x = bx \implies b(bx) = bx \implies b \cdot 1_R = 1_R$, so $b = 1_R$.

   (b) Applying the hint, we see that $(ua)(ua) = u(au)a = u \cdot 1_R \cdot a = ua$. Thus, by part (a), $ua = 1_R$. 