MATH 456-01

Solutions to Homework 5

Section 3.2

p. 62: 1, 2ab, 5, 7, 11, 13, (21), 23, 26

1. (a) \((a + b)(a - b) = aa + a(-b) + ba + b(-b) = a^2 - ab + ba - b^2\).
   
   (b) \((a+b)^3 = (a+b)(a+b)(a+b) = (aa+ab+ba+bb)(a+b) = a^3+aa^2b+a^2b+ab^2+bab+bba+abb+bbb =
   
   a^3 + a^2b + aba + ab^2 + ba^2 + bab + b^2a + b^3\).
   
   (c) (a) becomes \(a^2 - b^2\) and (b) becomes \(a^3 + 3a^2b + 3ab^2 + b^3\).

2. (a) \[
\begin{bmatrix}
0 & 0 \\
1 & 0 \\
0 & 0
\end{bmatrix}
\] and \[
\begin{bmatrix}
1 & 0 \\
0 & 1 \\
0 & 1
\end{bmatrix}
\] are easily seen to be idempotent. A little experimentation will yield \[
\begin{bmatrix}
0 & 0 \\
0 & 0 \\
0 & 0
\end{bmatrix}
\] as well.

(b) In \(\mathbb{Z}_{12}\), \(0^2 = 0, 1^2 = 1, 4^2 = 4\), and \(9^2 = 9\).

5. (a) \(S \cap T\) is a subring of \(R\). If \(a, b \in S \cap T\), then \(a, b \in S \implies a - b, ab \in S\) and \(a, b \in T \implies a - b, ab \in T\). Thus \(S \cap T\) passes the two-step subring test.

(b) \(S \cap T\) need not be a subring of \(R\); for example, \(S = \{0, 2, 4\}\) and \(T = \{0, 3\}\) are both subrings (check), but \(S \cup T = \{0, 2, 3, 4\}\) is not closed under addition.

7. (a) \(S = \{0, 2, 4, 6, 8\}\). It passes the two-step subring test, so it is a subring of \(\mathbb{Z}_{10}\).

(b) If \(A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\), then \(AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} b & 0 \\ 0 & 0 \end{bmatrix}\). For this to be the zero matrix, we need \(b = d = 0\). Thus, the members of \(S\) are of the form \(\begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix}\). If \(A_1, A_2 \in S\), then

\[
A_1 - A_2 \in S \quad \text{as well.} \quad A_1A_2 = \begin{bmatrix} a_1 & 0 \\ c_1 & 0 \end{bmatrix} \begin{bmatrix} a_2 & 0 \\ c_2 & 0 \end{bmatrix} = \begin{bmatrix} a_1a_2 & 0 \\ c_1c_2 & 0 \end{bmatrix} \in S. \quad \text{Therefore,} \quad S \text{ is a subring of} \ R.
\]

(c) If \(r_1, r_2 \in S\), then \((r_1 - r_2)b = r_1b - r_2b = 0_R - 0_R = 0_R\), so \(r_1 - r_2 \in S\). Also, \((r_1r_2)b = r_1(r_2b) = r_1(0_R) = 0_R\), so \(r_1r_2 \in S\). Therefore \(S\) is a subring of \(R\). (\(S\) is called the annihilator of \(b\).)

11. (a) To prove that \(ab\) is a unit, we will just show that it has an inverse: \((ab)(b^{-1}a^{-1}) = a(bb^{-1})a^{-1} = a(1_Ra^{-1}) = aa^{-1} = 1_R\). Similarly, \((b^{-1}a^{-1})(ab) = 1_R\).

(b) If the ring is commutative, then \(a^{-1}b^{-1} = b^{-1}a^{-1}\), so \(a^{-1}b^{-1}\) would be the multiplicative inverse of \(ab\). To find the example we seek, we need to look in a non-commutative ring like \(M_{2 \times 2}(\mathbb{Z})\).

Consider \(a = \begin{bmatrix} 1 & 4 \\ 1 & 5 \end{bmatrix}\), \(b = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}\). \(a^{-1} = \begin{bmatrix} 5 & -4 \\ -1 & 1 \end{bmatrix}\) and \(b^{-1} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}\). We get \(ab = \begin{bmatrix} 6 & 11 \\ 7 & 13 \end{bmatrix}\), so \((ab)^{-1} = \begin{bmatrix} 13 & -11 \\ -7 & 6 \end{bmatrix}\). On the other hand, \(a^{-1}b^{-1} = \begin{bmatrix} 14 & -23 \\ -3 & 5 \end{bmatrix} \neq (ab)^{-1}\).

13. It is not possible. Suppose \(x\) is a unit in a ring \(R\) with identity \(1_R\) and \(xr = 0\) for some \(r \in R\). Then there is an element \(u \in R\) such that \(ux = 1_R\), so \(uxr = u \cdot 0 \implies 1_Rr = 0 \implies r = 0\). Thus \(x\) fails condition (2) of the definition of a zero divisor.

23. We already know that if \(R\) is an integral domain, then cancellation holds in \(R\). For the converse, suppose that cancellation holds in \(R\) and that there exist \(a, b \in R\) such that \(a \neq 0\) but \(ab = 0\). Since \(a \cdot 0 = 0\), we have \(ab = a \cdot 0\), so cancellation implies \(b = 0\). Therefore, \(R\) is an integral domain.

26. We can start by inserting the products with \(0_R\) and \(1_R\):
By Theorem 3.8, $ax = a$ has a unique solution, so $a$ cannot appear again in the $a$ row of the table. Similarly, there cannot be another $b$ in the $b$ column of the table (since the solution of $yb = b$ is unique). Thus $ab = 1_R$ and the rest of the table practically fills itself in!

<table>
<thead>
<tr>
<th></th>
<th>$1_R$</th>
<th>$a$</th>
<th>$b$</th>
</tr>
</thead>
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<tr>
<td>$0_R$</td>
<td>$0_R$</td>
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<tr>
<td>$1_R$</td>
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</tr>
<tr>
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</tr>
<tr>
<td>$b$</td>
<td>$0_R$</td>
<td>$b$</td>
<td>$1_R$</td>
</tr>
</tbody>
</table>