2. Remember here that $x^2 + 1 = 0$, or $x^2 = 2$.

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<td>x</td>
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<td>2x</td>
<td>2x + 1</td>
</tr>
</tbody>
</table>

And if you don’t think that was a major pain, you are mistaken!

For Exercises 5-8, we can choose linear (or constant) representatives for each equivalence class by Corollary 5.5 since each polynomial is quadratic. Notice that in each ring, we have $([a + c]x + [b + d]) = ([a + c]x + [b + d])$, so we only need a rule for multiplication.

5. In this ring, $x^2 = -1$. We have $[ax + b][cx + d] = [ax^2 + (ad + bc)x + bd] = [(ad + bc)x + (bd - ac)]$.

6. In this ring, $x^2 = 2$, so we get $[ax + b][cx + d] = [ax^2 + (ad + bc)x + bd] = [(ad + bc)x + (2ac + bd)]$.

7. In this ring, $x^2 = 3$, so we get $[ax + b][cx + d] = [ax^2 + (ad + bc)x + bd] = [(ad + bc)x + (3ac + bd)]$.

8. In this ring, $x^2 = 0$, so we get $[ax + b][cx + d] = [ax^2 + (ad + bc)x + bd] = [(ad + bc)x + bd]$.

11. $\mathbb{Q}[x]/(x^2)$ is not a field; in fact, it is not even an integral domain: $[x]$ is a zero divisor! $[x][x] = [x^2] = [0]$.

14. (a) $[2x - 3]$ is a unit because $[2x - 3]$ is relatively prime to $(x^2 - 2)$. Suppose its inverse is $[ax + b]$. Then $2x - 3 a x + b = [2b - 3a] x + [4a - 3b]$ (using the formula from Exercise 6). Thus, we need $2b - 3a = 0$ and $4a - 3b = 1$. Solving gives $a = -2$ and $b = -3$, so $[2x - 3]^{-1} = [-2x - 3]$.

(b) $[f(x)] = [x^2 + x + 1] = [(x^2 + 1) + x] = [x]$ in this ring. Since $x$ is relatively prime to $x^2 + 1$ in $\mathbb{Z}_2[x]$, Theorem 5.9 implies that $[x]$ is a unit in $\mathbb{Z}_2[x]/(x^2 + 1)$. Suppose that its inverse is $[ax + b]$. Then $[x](ax + b) = [bx - a]$ (using the formula from Exercise 5). Thus, we need $a = -1$ and $b = 0$, so $[x]^{-1} = [-x]$.

(c) Since $x^3 + x + 1$ is irreducible over $\mathbb{Z}_2$ and $\deg(x^2 + x + 1) < \deg(x^3 + x + 1)$, the two are relatively prime. Thus, again by Corollary 5.5, $[x^2 + x + 1]$ is a unit in this ring. I will take advantage of the
fact that $x^3 = -x - 1 = x + 1$ in this ring; this also implies that $x^4 = x \cdot x^3 = x(x + 1) = x^2 + x$.

Suppose the inverse is $[ax^2 + b + c]$. Then

$$[x^2 + x + 1][ax^2 + bx + c] = [ax^4 + (a + b)x^3 + (a + b + c)x^2 + (b + c)x + c]$$

$$= [a(x^2 + x) + (a + b)(x + 1) + (a + b + c)x^2 + (b + c)x + c]$$

$$= [(2a + b + c)x^2 + (2a + 2b + c)x + (a + b + c)]$$

Thus, we need $b + c = 0$, $c = 0$, and $a + b + c = 1$ in $\mathbb{Z}_2$. $c = 0 \implies b = 0$ and $a = 1$, so our inverse is just $[x^2]$. 
