1. If \( p(x) \) and \( q(x) \) are constant polynomials, so are \( p(x) - q(x) \) and \( p(x)q(x) \); thus, \( K \) is a subring. However, since \( x \cdot 1 = x \notin K \) even though \( 1 \in K \), \( K \) is not an ideal.

3. (a) Let \((k,0),(l,0) \in I\). Then \((k,0) - (l,0) = (k - l,0) \in I\) and \((k,0)(l,0) = (kl,0) \in I\). Therefore, \( I \) is an ideal of \( \mathbb{Z} \times \mathbb{Z} \).

   (b) Since \((1,1) \in T\) but \((1,2)(1,1) = (1,2) \notin T\), \( T \) is not an ideal.

7. (a) \( (0) = \{0\}, (1) = \mathbb{Z}_5 = \{2\} = (3) = (4) \). (This illustrates a general principle: if \( F \) is a field, then the only ideals of \( F \) are \{0\} and \( F \).

   (b) \( (0) = \{0\}, (1) = \mathbb{Z}_9 = \{2\} = (4) = (5) = (7) = (8), (3) = (6) = \{0,3,6\} \).

   (c) \( (0) = \{0\}, (1) = \mathbb{Z}_{12} = (5) = (7) = (11), (2) = (10) = \{0,2,4,6,8,10\}, (3) = (9) = \{0,3,6,9\}, (4) = (8) = \{0,4,8\}, (6) = \{0,6\} \). Notice that \( (m) = (\gcd(m,12)) \).

12. Notice that if \( m \in (n) \), then \( (m) \subseteq (n) \) by closure.

   (a) For each \( m \in \mathbb{Z} \), \( (m) = (-m) \). Thus \( (1) = \mathbb{Z} = (-1) \) even though \( 1 \neq -1 \).

   (b) We did this one in class.

   (c) Since \( 6,9,15 \in (3), (6,9,15) \subseteq (3) \). Since \( 9 - 6 = 3,3 \in (6,9,15) \), so \( (3) \subseteq (6,9,15) \). Thus \( (3) = (6,9,15) \).

15. (a) If \( a, b \in I \cap J \), then \( a, b \in I \) and \( a, b \in J \). Thus \( a - b \in I \) and \( a - b \in J \) since \( I \) and \( J \) are ideals, so \( a - b \in I \cap J \). If \( r \in R \), then \( ra \in I \) and \( rb \in J \) since \( I \) and \( J \) are ideals, so \( ra \in I \cap J \). Therefore, \( I \cap J \) is an ideal.

   (b) Let \( a, b \in \bigcap_k I_k \), and let \( r \in R \). Then \( a, b \in I_k \) for each \( k \), so \( a - b \in I_k \) and \( ra \in I_k \) for each \( k \) since \( I_k \) is an ideal. Therefore, \( a - b, ra \in \bigcap_k I_k \), so \( \bigcap_k I_k \) is an ideal.

16. \( (2) \cup (3) \) is not a subring since \( 2 + 3 = 5 \) is in neither \( (2) \) nor \( (3) \).

17. Let \( a, b \in I \cap S \) and let \( r \in S \). Then \( a, b \in I \) and \( a, b \in S \), so \( a - b \in I \) and \( a - b \in S \). Thus \( a - b \in I \cap S \). Also, since \( I \) is an ideal, \( ra \in I \). Since \( S \) is a subring and \( r, a \in S \), \( ra \in S \), too. Therefore, \( ra \in I \cap S \), so \( I \) is an ideal of \( S \).

18. Since \( 0 \in I \) and \( 0 \in J \), \( I, J \subseteq I + J \). Now let \( x, y \in I + J, r \in R \). Then \( x = a + b \) and \( y = c + d \) for some \( a, c \in I \) and \( b, d \in J \). Thus \( x - y = (a - c) + (b - d) \in I + J \) since \( a - c \in I \) and \( b - d \in J \) by closure of \( I \) and \( J \) under subtraction. Also, \( rx = r(a + b) = ra + rb \in I + J \) since \( ra \in I \) and \( rb \in J \) (because \( I \) and \( J \) are ideals).

27. We know from exercise 15 that \( (m) \cap (n) \) is an ideal. Suppose that \( d \in (m) \cap (n) \). Then \( m|d \) and \( n|d \). Since \( m \) and \( n \) are relatively prime, this implies that \( mn|d \). (See exercise 17 of Section 1.2.) Thus \( d \in (mn) \), so \( (m) \cap (n) \subseteq (mn) \). Conversely, if \( d \in (mn) \), then \( mn|d \), so \( m|d \) and \( n|d \). Therefore, \( d \in (m) \cap (n) \). This gives us \( (m) \cap (n) = (mn) \) if \( (m,n) = 1 \).

35. This is the converse of my remark in 7(a). Let \( a \in R, a \neq 0_R \). Then \( a \in (a) \), so \( (a) \neq (0_R) \). Thus, by assumption, \( (a) = R \). Since \( R \) has a unity \( 1_R \), \( 1_R \in (a) \). That is, there exists \( b \in R \) such that \( ab = 1_R \). Therefore, \( a \) has a multiplicative inverse in \( R \). Since we already know that \( R \) is a commutative ring with identity, \( R \) is in fact a field.