2. Let \( \phi : F \rightarrow R \) be a homomorphism, and let \( S \) be the image of \( \phi \). Let \( K \) be the kernel of \( \phi \). If \( S = \{0_R\} \), then we are done. Assume that \( S \neq \{0_R\} \). Since \( \ker \phi \) is an ideal and \( F \) is a field, the kernel is either \( \{0_F\} \) or \( F \) itself. However, if the kernel is \( F \) itself, then every element of \( F \) is mapped to \( 0_R \); i.e., the image of \( \phi \) would be \( \{0_R\} \). Since this is not the case, the kernel must be \( \{0_F\} \), so \( \phi \) is one-to-one. Now \( F/\ker \phi \cong S \) by the first isomorphism theorem. But since \( F \cong F/\{0_F\} \), we have \( F \cong S \), as desired.

5. This need not be true. For example, \( \mathbb{Z} \) is an integral domain and \((6)\) is an ideal of \( \mathbb{Z} \), but \( \mathbb{Z}/(6) \cong \mathbb{Z}/6 \) is not an integral domain.

7. (a) \( T \) is a subring of \( \mathbb{Z} \) and \( I = (6) \) is an ideal in \( \mathbb{Z} \), so by Exercise 17 of 6.1, \( I \) is an ideal in \( T \). (You could also prove this directly.)

(b) The elements of \( T/I \) are \( 0 + I \) and \( 3 + I \) since \( 6 + I = 0 + I \), etc. Here are the tables:

<table>
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<tr>
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<th>( 0 + I )</th>
<th>( 3 + I )</th>
<th>( 0 + I )</th>
<th>( 3 + I )</th>
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<tr>
<td>( 0 + I )</td>
<td>( 0 + I )</td>
<td>( 3 + I )</td>
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<td>( 3 + I )</td>
<td>( 3 + I )</td>
<td>( 0 + I )</td>
<td>( 3 + I )</td>
<td></td>
</tr>
</tbody>
</table>

The multiplicative identity is \( 3 + I \). The only nonzero element is \( 3 + I \), and it has an inverse (itself). Thus, \( T/I \) is a field. Notice that it is isomorphic to \( \mathbb{Z}_2 \).

9. Define \( \phi : R \rightarrow R \) by \( \phi(r) = r \) for all \( r \in R \). \( \phi \) is clearly surjective and \( \ker \phi = \{0_R\} \), so by the first isomorphism theorem, \( R/(0_R) \cong R \).

14. Let \( x + I, y + I \in R/I \). Then \( xy - yx \in I \), so \( xy - yx = i \) for some \( i \in I \). Thus \( xy = yx + i \), so \( xy \notin yx + I \). We also know that \( xy \in yx + I \). Since the cosets \( xy + I \) and \( yx + I \) are not disjoint, they must be equal. Now \( (x + I)(y + I) = (xy + I) \), so \( R/I \) is commutative.

23. Define \( \phi : \mathbb{Z}_{20} \rightarrow \mathbb{Z}_5 \) by \( \phi([n]_{20}) = [n]_5 \). We must show first that \( \phi \) is well-defined: if \( [m]_{20} = [n]_{20} \), then \( 20|m - n \), so \( 5|m - n \) as well. Thus \( [m]_5 = [n]_5 \), and \( \phi([m]_{20}) = \phi([n]_{20}) \). \( \phi \) is also a surjective homomorphism: if \( [n]_5 \in \mathbb{Z}_5 \), then \( \phi([n]_{20}) = [n]_5 \), and \( \phi([m]_{20} + [n]_{20}) = \phi([m + n]_{20}) = [m + n]_5 \). \( \phi \) is a well-defined homomorphism. Finally, if \( \phi([n]_{20}) = [0]_5 \), then \( [n]_5 = [0]_5 \), so \( \phi([n]_{20}) = [5k]_{20} \) for some \( k \in \mathbb{Z} \). Thus \( \ker \phi \subseteq ([5]_{20}) \). Conversely, if \( [n]_{20} \in ([5]_{20}) \), then \( [n]_{20} = [5k]_{20} \), so \( \phi([n]_{20}) = [5k]_5 = [0]_5 \). Therefore, \( \ker \phi = ([5]_{20}) \).

By the First Isomorphism Theorem, \( \mathbb{Z}_{20}/([5]_{20}) \cong \mathbb{Z}_5 \).

For Exercises 28-31, I will find a homomorphism between the given rings whose kernel is \( I \).

28. Define \( \phi : S \rightarrow \mathbb{Z}_2 \) by \( \phi([m]/n) = [m]_2 \), where \( m/n \) is in lowest terms. Then \( \phi \) is well-defined since only one representative for each fraction is in lowest terms. (That is, if \( a/b = c/d \) with \( a/b \) and \( c/d \) in lowest terms, then \( a = c \) and \( b = d \).) Clearly \( \phi \) is surjective. Let \( a/b, c/d \in S \), and let \( q = \gcd(b, d) \).

Then \( \phi(a/b + c/d) = \phi\left(\frac{ad' + bd'}{bdq}\right) = [ad' + bd'] = [a] + [c] = \phi(a/b) + \phi(c/d) \).

Recall that \( b \) and \( d \) are odd by the definition of \( S \), so \( [b] = [d'] = [d] \). (Before we could apply \( \phi \), we needed to get the gcd out of there so that our fraction was in lowest terms). We also get \( \phi\left(\frac{a}{b}\right) = \phi\left(\frac{ac}{bd}\right) = [ac] = [a] + [c] = \phi(a/b) + \phi(c/d) \).

Thus, \( \phi \) is a homomorphism. Finally, \( \ker \phi = I \) (\( \phi([m]/n) = [0]_2 \) \( \Leftrightarrow \) \( m \) is even), so \( S/I \cong \mathbb{Z}_2 \) by the First Isomorphism Theorem.

29. This is a generalization of Exercise 28. Please excuse any cut-and-paste errors.

Define \( \phi : T \rightarrow \mathbb{Z}_p \) by \( \phi(m/n) = [m \cdot n^{-1}]_p \), where \( m/n \) is in lowest terms and \( n^{-1} \) is the multiplicative inverse of \( n \) mod \( p \), which exists since \( (p, n) = 1 \). Then \( \phi \) is well-defined since only one representative for
Define. This is a generalization of Exercise 30. Please excuse any cut-and-paste errors.

31. Define $\phi : T \to \mathbb{R}$ by $\phi \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = a$. $\phi$ is clearly surjective. (If $a \in \mathbb{R}$, then $\phi \left( \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \right) = a$.) $\phi$ is also well-defined since each such matrix has a unique representation.

We also get $\phi \left( \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \right) = \phi \left( \begin{bmatrix} ac & ad + bc \\ 0 & ac \end{bmatrix} \right)$, so the kernel is $I$. Thus, by the First Isomorphism Theorem, $T/I \cong \mathbb{R}$. 

32. This is a generalization of Exercise 30. Please excuse any cut-and-paste errors.

Define $\phi : T \to \mathbb{R} \times \mathbb{R}$ by $\phi \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = (a, c)$. $\phi$ is clearly surjective. (If $(x, y) \in \mathbb{R}$, then $\phi \left( \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \right) = (x, y)$.) $\phi$ is also well-defined since each such matrix has a unique representation.

We also get $\phi \left( \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \right) = \phi \left( \begin{bmatrix} ac & ad + bf \\ 0 & cf \end{bmatrix} \right)$, so the kernel is $I$. Thus, by the First Isomorphism Theorem, $S/I \cong \mathbb{R} \times \mathbb{R}$. 

33. This is a generalization of Exercise 30. Please excuse any cut-and-paste errors.