1. If \( n \) is composite, then \( n = ab \) for some \( a, b \in \mathbb{Z}, 1 < a, b < n \). Thus \( ab = n \in (n) \), but \( a \notin (n) \) and \( b \notin (n) \) (since \( n \) divides neither \( a \) nor \( b \)).

2. By Exercise 27 in 6.1, since \( (2, 3) = 1, (2) \cap (3) = (6) \). But \( (2) \) and \( (3) \) are prime ideals, while \( (6) \) is not.

3. (a) Suppose first that \( p \) is prime and \( I \) is an ideal of \( \mathbb{Z} \) such that \( (p) \subseteq I \subseteq \mathbb{Z} \). If \((p) \neq I \), then there exists \( n \in I - (p) \). Such an \( n \) is not a multiple of \( p \), so, since \( p \) is prime, \((n, p) = 1 \). Therefore, there are integers \( x \) and \( y \) such that \( nx + py = 1 \). But \( n, p \in I \), so \( nx + py \in I \). That is, \( I = \mathbb{Z} \). Therefore, \( I \) is maximal.

Conversely, suppose that \( (p) \) is a maximal ideal of \( \mathbb{Z} \). If \( p = ab \), then \((p) \subseteq (a) \subseteq \mathbb{Z} \), so \((a) = (p) \) or \((a) = \mathbb{Z} \). In the first case, \( p/a \) and \( a/p \), so \( a = \pm p \). In the second case, \((a) = \pm 1 \). Thus, in either case, we have a trivial factorization of \( p \), so \( p \) is prime.

Here is the easy way: \( p \) is prime if and only if \( \mathbb{Z}/(p) \cong \mathbb{Z}_p \) is an integral domain, which is true if and only if \( \mathbb{Z}_p \) is finite, \( 1/p \) is false if and only if \( \mathbb{Z}_p \) is a field, which is true if and only of \( (p) \) is maximal.

(b) Suppose first that \( p(x) \) is prime and \( I \) is an ideal of \( F[x] \) such that \((p(x)) \subseteq I \subseteq F[x] \). If \((p(x)) \neq I \), then there exists \( n(x) \in I - (p(x)) \). Such an \( n(x) \) is not a multiple of \( p(x) \), so, since \( p(x) \) is irreducible, \( (n(x), p(x)) = 1 \). Therefore, there are polynomials \( g(x) \) and \( h(x) \) such that \( n(x)g(x) + p(x)h(x) = 1 \). But \( n(x), p(x) \in I \), so \( n(x)g(x) + p(x)h(x) \in I \). That is, \( I = F[x] \). Therefore, \( I \) is maximal.

Conversely, suppose that \((p(x)) \) is a maximal ideal of \( F[x] \). If \( p(x) = a(x)b(x) \), then \((p(x)) \subseteq (a(x)) \subseteq \mathbb{Z} \), so \((a(x)) = (p(x)) \) or \((a(x)) = F[x] \). In the first case, \( p(x)a(x) \) and \( a(x)p(x) \), so \( a(x) \) is an associate of \( p(x) \). In the second case, \((a(x)) = (p(x)) \) is a constant polynomial. Thus, in either case, we have a trivial factorization of \( p(x) \), so \( p(x) \) is prime.

4. If \( R \) is an integral domain and \( ab \in (0) \), then \( ab = 0 \). Thus \( a = 0 \) or \( b = 0 \), so \( a \in (0) \) or \( b \in (0) \). Therefore, \((0) \) is a prime ideal. On the other hand, if \((0) \) is a prime ideal and \( ab \in (0) \), then \( ab \in (0) \), so \( a \in (0) \) or \( b \in (0) \). Thus \( a = 0 \) or \( b = 0 \), so \( R \) is an integral domain.

5. The ideals in \( \mathbb{Z}_6 \) are \( (0), (1) = (5), (2) = (4), \) and \( (3) \). Of these, \( (2) = (4) \) and \( (3) \) are maximal.

The ideals in \( \mathbb{Z}_{12} \) are \( (0), (1) = (5) = (7) = (11), (2) = (10), (3) = (9), (4) = (8), \) and \( (6) \). Of these \( (2) = (10) \) and \( (3) = (9) \) are maximal.

6. Let \( a \in F, a \neq 0 \). If \( (0) \) is maximal, then \( (0, a) = F \), so there are elements \( b, c \in F \) such that \( 0(b) + a(c) = 1 \). Thus \( ac = 1 \), so \( a \) has an inverse. Since \( a \) was arbitrary, \( F \) is a field. Conversely, if \( F \) is a field, then the only ideals are \( (0) \) and \( F \), so \((0) \) is maximal.

11. Define \( \phi \) from \( \mathbb{Z}[x] \) to \( \mathbb{Z} \) by \( \phi(p(x)) = p(1) \). We have seen several times that evaluation is a homomorphism, and this one is surjective. Certainly \( x - 1 \in \ker \phi \). If \( \frac{f(x)}{x - 1} \in \ker \phi \), then \( f(x) = (x - 1)g(x) \) for some \( g(x) \in \mathbb{Z}[x] \) by the Factor Theorem and Theorem 4.22, so \( f(x) \in (x - 1) \). Therefore, \( \ker \phi = (x - 1) \). Thus \( \mathbb{Z}[x]/(x - 1) \cong \mathbb{Z} \). Since \( \mathbb{Z} \) is an integral domain but not a field, \( (x - 1) \) is a prime ideal but not a maximal ideal.

13. Define \( \phi \) from \( \mathbb{Z} \times \mathbb{Z} \) to \( \mathbb{Z} \) by \( \phi(m, n) = m \). Then \( \phi \) is a surjective homomorphism and \( \ker \phi = (0) \times \mathbb{Z} \), so \( \mathbb{Z} \times \mathbb{Z}/(0) \cong \mathbb{Z} \). Since \( \mathbb{Z} \) is an integral domain but not a field, \( 0 \times \mathbb{Z} \) is a prime ideal but not a maximal ideal. [Note the technique in Exercises 11 and 13 -- this is a way to come up with ideals with certain properties!]

16. \( M \) is clearly an ideal: if \( a, b \in M \), then \( a - b \in M \), and if \( a \in M \) and \( r \in \mathbb{Z} \), then \( ar \in M \). If \( M \subseteq J \subseteq \mathbb{Z} \) but \( J \neq M \), then there is an element \( a \in J - M \). Thus \( a \) is not a multiple of \( 4 \), but \( a \) is even, so \( a = 4q + 2 \) for some integer \( q \). Thus \( 4q - a = 2 \in J \) since \( 4 \in J \), so \( J = 2\mathbb{Z} \). Therefore, \( M \) is maximal.