1. (a) $f(x + y) = 3(x + y) = 3x + 3y = f(x) + f(y)$. Certainly $< 4 \leq \ker f$. If $f(x) = 0$, then $3x = 0$ in $\mathbb{Z}_{12}$, so $4|x$. Thus $\ker f = < 4 >$. (b) $f(k + 1) = ([k + l]_2, [k + l]_4) = ([k]_2, [k]_4) + ([l]_2, [l]_4) = f(k) + f(l)$, so $f$ is a homomorphism. If $k \in \ker f$, then $k$ is congruent to $0$ mod $2$ and mod $4$, so $4|k$. Conversely, if $4|k$, then $k \in \ker f$. Thus $\ker f = < 4 >$. (c) We have seen several times that this kind of map is operation-preserving. $[k]_8 \in \ker f$ if and only if $[k]_2 = [0]_2$, or $2|k$. Thus $\ker f = \{0, 2, 4, 6\}$ in $\mathbb{Z}_8$. (d) We must check that $\phi(fg) = \phi(f)\phi(g)$. Notice that since $fg \in S_n$, $\phi(fg)(k) = fg(k)$ if $1 \leq k \leq n$ and $n + 1$ if $k = n + 1$. On the other hand, $\phi(f)\phi(g)(k) = \phi(f)(g(k))$ if $k < n + 1$ and $\phi(f(n + 1))$ if $k = n + 1$. In the first case, since $g(k) < n + 1$, $\phi(f)(g(k)) = f(g(k)) = fg(k)$. In the second case, $\phi(f(n + 1)) = n + 1$. In both cases, we get the same result as we did for $\phi(fg)$, so $\phi$ is a homomorphism. Its kernel is just $\{e\}$ since no other permutation in $S_n$ fixes every element. (e) $h(x + y) = 2(x + y) = 2x + 2y = h(x) + h(y)$, where the computations are being performed in the appropriate groups. Thus $h$ is a homomorphism. $x \in \ker h$ if and only if $h(x) = [2x]_3 = [0]_3$. Since $3|2x$ if and only if $3|x$, we have $\ker h = < [3]_3 >$.

3. (a) This is Exercise 7 in 7.6. (b) Define $\phi : G \to G^*$ by $\phi(g) = (g, e_H)$. Then $\phi$ is clearly bijective, and $\phi(gg') = (gg', e_H) = (g, e_H)(g', e_H) = \phi(g)\phi(g')$, so $\phi$ is also a homomorphism. The proof for $H$ is similar. (c) Define $\phi : G \times H \to H$ by $\phi(g, h) = h$. $\phi$ is clearly surjective, and $(g, h) \in \ker \phi$ if and only if $h = \phi(g, h) = e$. Thus $\ker \phi = G^*$, so $G \times H/G^* \cong H$. The proof of the other part is similar.

5. (a) The kernels of such homomorphisms have orders dividing $12$, so the images must also have orders dividing $12$. Since $\mathbb{Z}_{12}$ is cyclic, so are all of its quotient groups and hence, by the First Isomorphism Theorem, its homomorphic images are, too. Thus the images must be cyclic groups of orders dividing $12$. Since cyclic groups of a given order are unique up to isomorphism, our list is $\{\}$, $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_4$, $\mathbb{Z}_5$, $\mathbb{Z}_{10}$, and $\mathbb{Z}_{20}$. (b) Reasoning as in (a), we see that the possible homomorphic images of $\mathbb{Z}_{20}$ are $\{\}$, $\mathbb{Z}_2$, $\mathbb{Z}_4$, $\mathbb{Z}_5$, $\mathbb{Z}_{10}$, and $\mathbb{Z}_{20}$.

11. Define $\phi : \mathbb{R}^* \to \mathbb{R}^*$ by $\phi(x) = |x|$. We have already seen that this is a surjective homomorphism. $x \in \ker \phi$ if and only if $|x| = \phi(x) = 1$, so $x = \pm 1$. Thus $\ker \phi = \{1, -1\}$, so $\mathbb{R}^*/\{1, -1\} \cong \mathbb{R}^*$.

15. Define $\phi : GL(2, \mathbb{R}) \to \mathbb{R}^*$ by $\phi(M) = \det M$ for each $M \in GL(2, \mathbb{R})$. $\phi$ is well-defined since members of $GL(2, \mathbb{R})$ have nonzero determinant. It is also a homomorphism since $\det(MN) = \det(M)\det(N)$, and it is surjective since $\det \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} = a$. $M \in \ker \phi$ if and only if $\det M = 1$, which is precisely how $SL(2, \mathbb{R})$ is defined. Thus, by the First Isomorphism Theorem, $GL(2, \mathbb{R})/SL(2, \mathbb{R}) \cong \mathbb{R}^*$.

23. Following the hint, we see that $T$ is one-to-one since $T(f(x)) = T(g(x)) \Rightarrow Z + xf(x) = Z + xg(x)$, so $x(g(x) - f(x)) \in Z$. This can only happen if $g(x) = f(x)$. It is also surjective: if $f(x) + Z \in \mathbb{Z}[x]/Z$, then $f(x) = xg(x) + n$ for some $g(x) \in \mathbb{Z}[x], n \in Z$ by the division algorithm. Thus $f(x) + Z = xg(x) + Z$, and $T(g(x)) = xg(x) + Z = f(x) + Z$. Finally, $T$ is a homomorphism since $T(f(x) + g(x)) = Z + (f(x) + g(x)) = (Z + f(x)) + (Z + g(x)) = T(f(x)) + T(g(x))$. Therefore $\mathbb{Z}[x] \cong \mathbb{Z}[x]/Z$. 

Section 7.8

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