

# EMBEDDED TREE STRUCTURES AND EIGENVALUE STATISTICS OF GENUS ZERO ONE-FACE MAPS

DISSERTATION SUMMARY - ERIN MCNICHOLAS

My dissertation has focused on connections between random matrix theory and graph theory. This area of research examines matrices derived from random graphs, and involves extensive numerical simulations and combinatorics.

I began by studying the adjacency matrices of a special class of random three-regular graphs. These graphs can be decomposed into a cycle graph, and a permuted “toothpick” graph. The toothpick graph is the totally disjoint, one-regular graph in which vertex  $v_1$  is adjacent to vertex  $v_2$ ,  $v_3$  is adjacent to  $v_4$ , and so on. By permuting the vertex labels we generate random pairings between the  $2N$  vertices. The resulting permuted toothpick graph is still totally disjoint and one-regular. Superimposing a permuted toothpick graph on the cycle graph is equivalent to adding edges between random pairs of vertices in such a way that every vertex has exactly three incident edges. The adjacency matrix of the resulting three-regular graph can be expressed  $C + P^TTP$ , where  $C$  is the adjacency matrix of the cycle graph,  $T$  is the adjacency matrix of the toothpick graph, and  $P$  is a random permutation matrix.

Three-regular graphs of this form can be used to represent one-face maps. Our graphs describe a glueing of the edges of the  $2N$ -gon. If the permuted toothpick portion of our three-regular graph contains an edge connecting vertex  $i$  and vertex  $j$ , then edge  $i$  and edge  $j$  are glued together in the corresponding  $2N$ -gon. This glueing maps the  $2N$ -gon to a graph embedded in a Riemann surface of genus  $g \geq 0$ . Since the complement of this graph is homeomorphic to the unit disc, we say the map has “one face”.

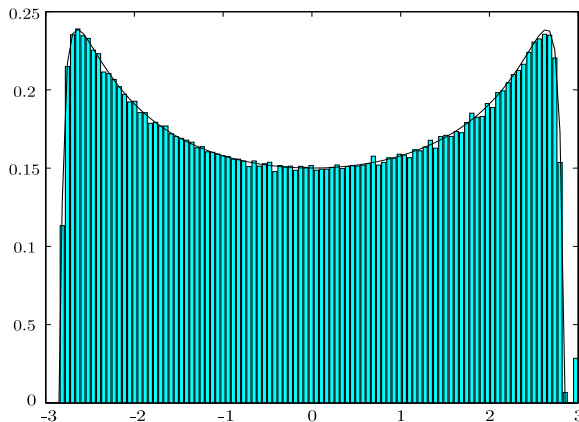


Figure 1: Empirical eigenvalue density for a random sample of 300 three-regular graphs of the form  $C + P^TTP$ , plotted with the McKay density function. The graphs used in this sample had 600 vertices each.

Numerical studies have revealed that the limiting eigenvalue statistics of our random three-regular graphs are the same as those of much larger, and more widely studied classes of random matrices. In particular, the limiting eigenvalue density is described by the McKay density, and the distribution of scaled eigenvalue spacings from the bulk of the spectrum appears to be that of the Gaussian Orthogonal Ensemble (GOE). Figure 1 shows the empirical eigenvalue density for a random sample of our three-regular graphs. Figure 2 shows the scaled eigenvalue spacing distribution

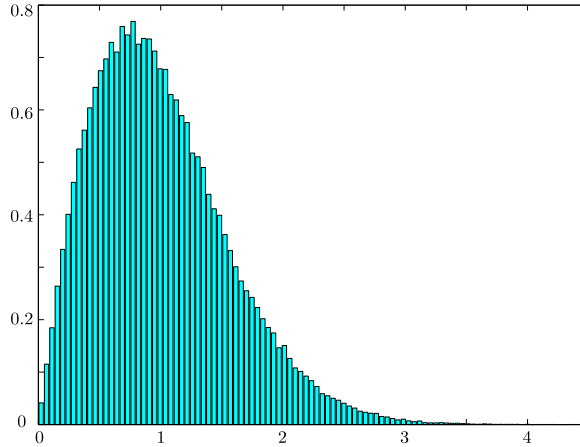


Figure 2: Distribution of scaled eigenvalue spacings over the bulk of the spectrum for a sample of 300 three-regular graphs of the form  $C + P^TTP$ . The graphs used in this sample had 600 vertices each.

over the bulk of the spectrum for the same sample.

The McKay density formula describes the limiting eigenvalue density for sequences of  $k$ -regular graphs in which the ratio of the number of  $r$ -cycles to the number of vertices tends to zero with probability one, for all values of  $r$ . In other words, McKay's formula applies to sequences of graphs which look like trees at most vertices. McKay proved that for a sequence of  $k$ -regular graphs, if the limiting eigenvalue density is not described by McKay's density formula, then the ratio of the number of  $r$ -cycles to the number of vertices must not approach zero, for some value of  $r$ .

The Gaussian Orthogonal Ensemble is the set of  $N \times N$  matrices whose components are independent (up to the symmetry requirement), random variables of mean zero and standard deviation one. Numerical studies have revealed that the distributions describing the eigenvalue statistics of the Gaussian Orthogonal Ensemble seem to describe the eigenvalue statistics of most real symmetric random matrices having independent entries. For certain classes of random symmetric matrices having independent entries, these results have been proven analytically. Since our random matrices represent three-regular graphs, the entries are highly dependent, and thus these universality results are not applicable.

A natural question is whether the eigenvalue statistics of our three-regular graphs depend on the genus of the underlying map. The vast majority of one-face maps having  $N$  edges in the embedded graph are of genus slightly less than  $\lfloor N/2 \rfloor$ . Only maps having genus near this upper bound will be sampled by a random selection of associated three-regular graphs. Thus, the McKay and GOE results for random samples of one-face maps having  $N$  edges in the embedded graph are only indicative of one-face maps having genera near  $\lfloor N/2 \rfloor$ . Using the one-to-one correspondence between non-crossing pair partitions and three-regular graphs representing genus zero maps, I have developed a recursive algorithm to generate random samples of genus zero one-face maps. This algorithm creates a non-crossing pair partition and then finds the adjacency matrix of the associated three-regular graph. Pairs are chosen according to a non-uniform probability mass function, ensuring that the corresponding genus zero one-face maps are sampled with equal probability. The generation of random one-face maps having specified genus greater than zero, and  $N$  edges remains an open problem.

My numerical studies of random three-regular graphs representing genus zero one-face maps have revealed many interesting properties of these graphs and their eigenvalue statistics. Figure 3 shows the eigenvalue density for a sample of random genus zero one-face maps. While numerical

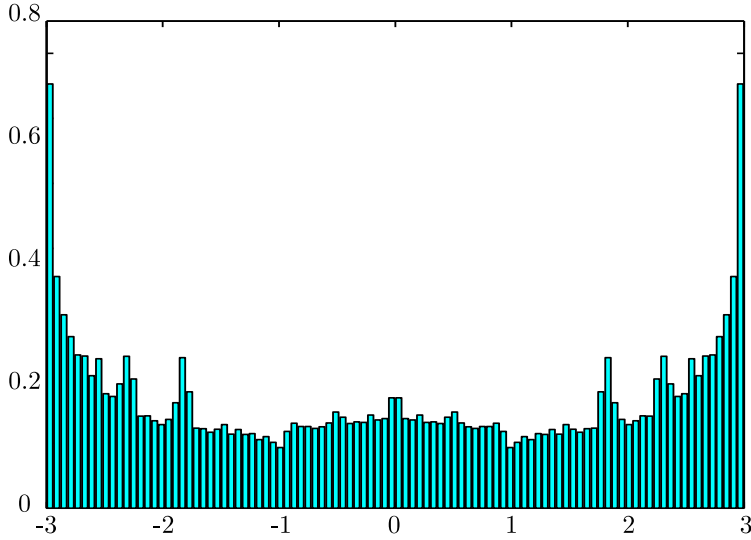


Figure 3: Eigenvalue density for a random sample of 100 three-regular graphs representing genus zero one-face maps. This sample contains graphs having 1000 vertices

experiments indicate that there is a limiting density formula in the genus zero case, it does not appear to be described by any established density function. Furthermore, the scaled eigenvalue spacings are described by the exponential distribution function, not the GOE spacing distribution, as shown in Figure 4. The graph of the  $j^{\text{th}}$  eigenvalue spacing as a function of  $j$  (Figure 5), has a very interesting form in the genus zero case.

Examining the adjacency matrices, we find the three-regular graphs representing genus zero one-face maps have a distinctive structure (Figure 6). The concentration of entries near the diagonal, and the existence of spines (series of entries forming anti-diagonal bands) can be understood given our construction algorithm. It is clear from the probability mass function used to generate the associated non-crossing pair partition that three-regular graphs representing genus zero maps will have a large number of double edges. Furthermore, they will have sets of edges adjoining vertex  $i$  to vertex  $j$ , vertex  $i + 1$  to vertex  $j - 1$ , and so on.

For one-face maps having small genus relative to  $N$ , a great deal of structure is imposed on the corresponding three-regular graph. I believe the striking difference between the eigenvalue statistics of genus zero maps and genus  $\sim \lfloor N/2 \rfloor$  is a result of this imposed structure. For any fixed genus, as the size of the map increases (i.e. the number of vertices in the associated three-regular graph grows), more structure will be imposed on the associated three-regular graph. This leads to the conjecture that for fixed genus, the limiting scaled spacing distribution will be exponential. For genus one maps, and smallish  $N$ , the numerical results are not inconsistent with this conjecture.

In the genus zero case, the one-face map's embedded graph is a planar tree and there is a correlation between its vertices and the primitive cycles of the associated three-regular graph. By understanding the probabilistic structure of these planar trees, we can better understand the structure of the three-regular graphs representing genus zero one-face maps.

Examining the adjacency matrices of the planar trees, I have found that under a natural labeling of the vertices, the adjacency matrix has a distinctive fan structure (see Figure 7). This structure is the result of our labeling convention, and certain properties of the planar tree; in particular the distribution of degree  $k$  vertices throughout the tree, and the number of degree  $k$  vertices adjacent to  $j$  degree  $m + 1$  vertices.

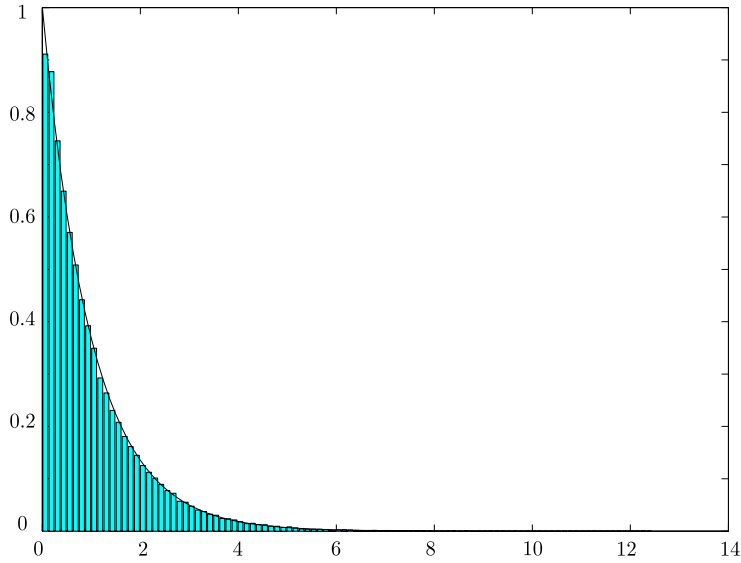


Figure 4: Distribution of scaled eigenvalue spacings for a sample of 100 three-regular graphs representing genus zero one-face maps, plotted with the exponential distribution function. This sample contains graphs having 1500 vertices.

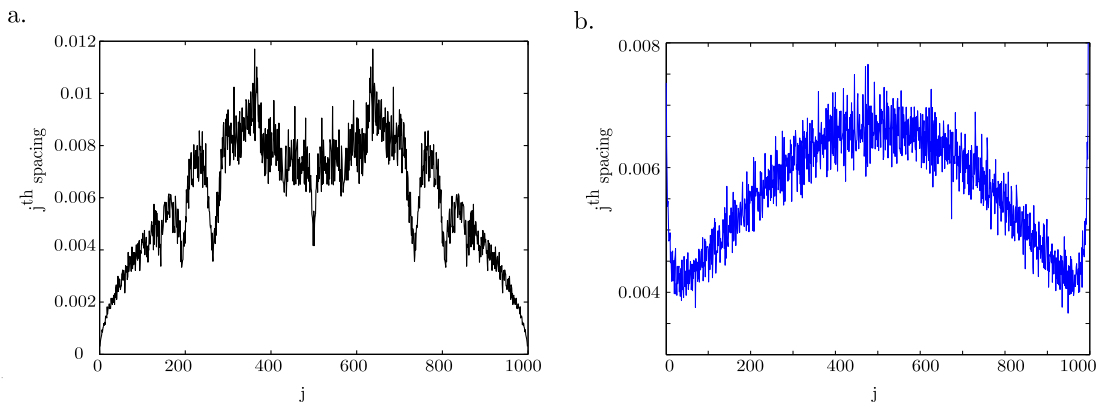


Figure 5: a. The  $j^{\text{th}}$  eigenvalue spacing as a function of  $j$  for a sample of 100 three-regular graphs representing genus zero one-face maps. b. The  $j^{\text{th}}$  eigenvalue spacing as a function of  $j$  for a sample of 100 three-regular graphs representing one-face maps of genera near  $g = 250$ . Both samples contain graphs having 1000 vertices.

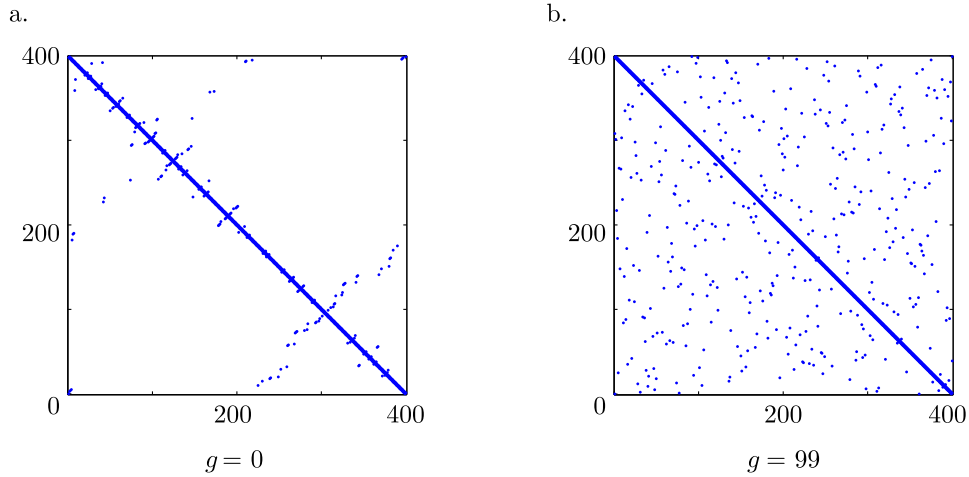


Figure 6: **Spy** images of random three-regular graphs of the form  $C + P^TTP$ . The graph in (a) represents a genus zero one-face map, while that in (b) represents a genus 99 one-face map. A dot in the **spy** image at the point  $(i, j)$  indicates the  $ij^{th}$  component of the adjacency matrix is non-zero.

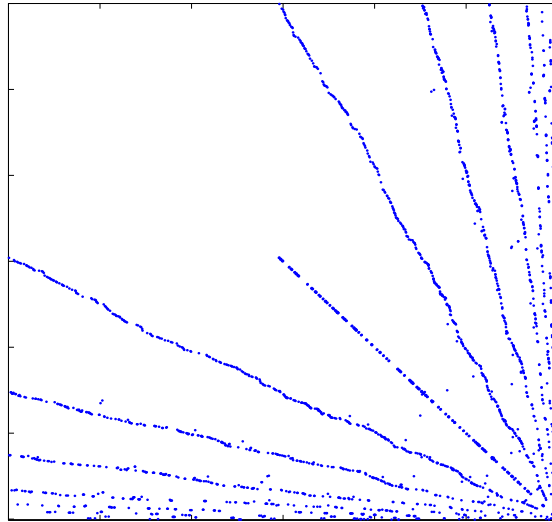


Figure 7: **Spy** image of the adjacency matrix of a random planar tree having 1200 vertices

Figure 8 shows the distribution of eigenvalues for a sample of random planar trees. Note the peaks at  $0, \pm 1, \pm\sqrt{2}$ . In a few instances, the eigenvalue  $\lambda = \frac{1+\sqrt{5}}{2}$  has been observed, sometimes with multiplicity greater than one. I have constructed subtrees that give eigenfunctions corresponding to  $\lambda = 0, \pm 1, \pm\sqrt{2}, \pm\sqrt{3}, \dots$ . For a given tree, the number of such eigenfunctions is again related to the distribution of vertex degrees and the expected number of degree  $k$  vertices adjacent to  $j$  degree  $m + 1$  vertices.

Every planar tree can be represented by a path on the integer lattice from  $(0, 0)$  to  $(2N, 0)$ , which may have several returns to the  $x$ -axis, but which never falls below it. Such a path is called a Dyck path. The Dyck path representation can be used to recast questions about the probabilistic structure of planar trees into questions about the number of Dyck paths having certain properties. The benefit of this Dyck path approach is that often complex questions about the structure of

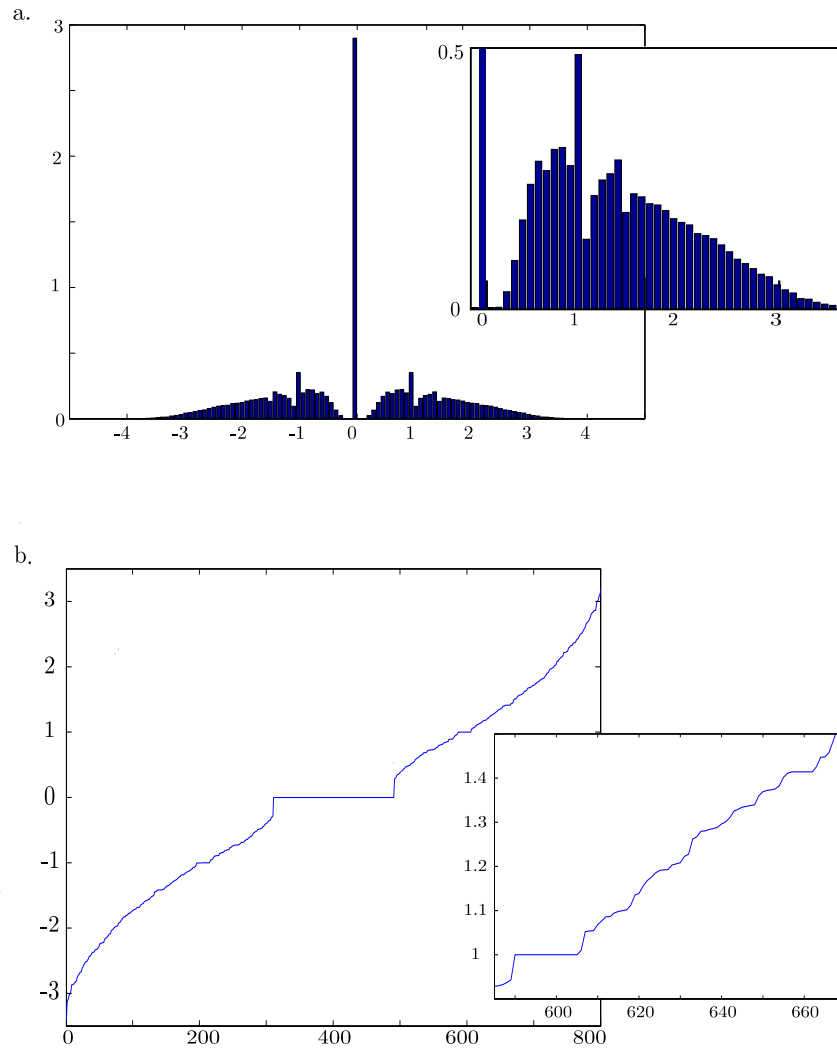


Figure 8: a. Eigenvalue density for a random sample of 100 planar trees having 800 edges each. b. A plot of the ordered eigenvalues for a random planar tree having 800 edges. Note the repeated eigenvalues at  $\lambda = 0, \pm 1$ , and  $\pm\sqrt{2}$ . Close ups of both figures are provided in the side boxes.

planar trees can be recast into straightforward counting problems. Using the Dyck path approach, I've found a simple combinatorial proof that the number of degree  $k$  vertices in a random planar tree is asymptotically  $\frac{N}{2k}$ . A different proof of this fact using generating functions was found by Drmota and Gittenberger. Degree  $k$  vertices correspond to length  $2k$  primitive cycles in the three-regular graphical representation of the genus zero one-face map. Thus, the  $\frac{N}{2k}$  asymptotic result verifies the fact that three-regular graphs corresponding to genus zero maps violate McKay's criteria and therefore the eigenvalue density is not described by McKay's density formula.

Using a complicated combinatorial argument, I proved that the percent of degree  $k$  vertices adjacent to  $j$  degree  $m + 1$  vertices is asymptotically

$$\binom{k-1}{j} \frac{1}{2^{k-1}} \tag{1}$$

when  $m = 0$ , and

$$\frac{1}{k} \binom{k}{j} \frac{(2^{m+1} - 1)^{k-j-1}}{2^{k(m+1)}} [2^{m+1}(k + j(m-1)) - km] \tag{2}$$

when  $m \geq 1$ . This result sheds light on both the structure of the planar tree and the three-regular graphical representation of the genus zero one-face map. If the planar tree contains a degree  $k$  vertex adjacent to  $j$  degree  $m + 1$  vertices, then the associated three-regular graph has a primitive cycle of length  $2k$  which shares  $j$  edges with  $j$  other primitive cycles, each having length  $2(m + 1)$ .

The Cantor-like appearance of repeated eigenvalues  $0, \pm 1, \pm\sqrt{2}, \dots$  suggest some sort of scaling property of the tree. Examining the distribution of vertex degrees in the tree after repeatedly deleting all degree one vertices, we find that after  $r$  rounds of deletions, the percent of degree  $k$  vertices is  $\frac{(r+1)^2}{(r+2)^k}$  for  $k \geq 2$ , and  $\frac{1}{(r+2)}$  for  $k = 1$ . Using the expected number of degree  $k$  adjacent to  $j$  degree  $m + 1$  results, we were able to prove the asymptotic result in the  $r = 1$  case.

Our study of genus zero one-face maps has revealed interesting properties and questions. Because these maps can be represented in a variety of ways (glueings of the  $2N$ -gon, three-regular graphs of the form  $C + P^TTP$ , non-crossing pair partitions, Dyck paths), there are many possible approaches when studying their structure. In particular, the three regular graphical representation and Dyck path approach have yielded a wealth of interesting results and avenues for future research.