

Group Exam 1

Good Luck To: _____

Math 457, Abstract Algebra II

And Good Luck To: _____

Fall 2009, Professor McNicholas

And Good Luck To: _____

Show your work and make sure your answers are well organized, easy to follow, and properly explained.

Problem 1:

a. proposition: If $G/Z(G)$ is cyclic then G is Abelian.

proof If $G/Z(G)$ is cyclic then there exists some generator $gZ(G)$ such that $\langle gZ(G) \rangle = G/Z(G)$. Let a, b be arbitrary elements of G . $aZ(G) = (gZ(G))^i = g^iZ(G)$ for some i , and $bZ(G) = (gZ(G))^j = g^jZ(G)$ for some j . $aZ(G) = g^iZ(G)$ implies $a = g^i z_1$ for some $z_1 \in Z(G)$. Similarly, $bZ(G) = g^jZ(G)$ implies $b = g^j z_2$ for some $z_2 \in Z(G)$. Thus $ab = (g^i z_1)(g^j z_2) = g^i g^j z_1 z_2$ since elements of the center commute with all elements of G . We have $ab = g^i g^j z_1 z_2 = g^{i+j} z_1 z_2 = g^{j+i} z_2 z_1 = g^j g^i z_2 z_1 = g^j z_2 g^i z_1 = ba$. Since a and b were arbitrary, we've shown the group operation on G is commutative. Thus G is Abelian.

■

proposition: If N is a normal subgroup of G and N and G/N are both p -groups, then G is a p -group.

proof Let x be an arbitrary element of G . Consider the coset xN in G/N . If G/N is a p -group, then $|xN| = p^k$ for some $k \in \mathbb{Z}^+$. In other words $(xN)^{p^k} = x^{p^k}N = N$. By properties of cosets, this implies $x^{p^k} \in N$. If N is also a p -group, then every element of N has order equal to some power of p . Thus $|x^{p^k}| = p^j$, i.e. $(x^{p^k})^{p^j} = x^{p^{k+j}} = e$. This implies $|x| \mid p^{k+j}$, which implies $|x| = p^i$ for some positive integer i . Thus, since x was arbitrary, the order of every element in G is equal to p to some power. By definition of p -group, this implies G is a p -group. ■

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Problem 2:

a. Use the class equation to prove that if $|G| = p^k$ for some prime p , then $|Z(G)| = p^l$ for some integer $l \geq 1$.

proof The class equation states $|G| = |Z(G)| + |cl(a_1)| + \dots + |cl(a_n)|$ where the $cl(a_i)$ are distinct conjugacy classes having more than one element (i.e. $a_i \notin Z(G)$). The class equation can also be written as $|G| = |Z(G)| + [G : C(a_1)] + \dots + [G : C(a_n)]$. For each $a_i \in G$, $[G : C(a_i)] \mid |G|$. Furthermore, if $a_i \notin Z(G)$, $[G : C(a_i)] > 1$. Thus, since $|G| = p^k$, each term of the form $[G : C(a_i)]$ in the class equation is divisible by p . $|G|$ is divisible by p , thus $p \mid (|Z(G)| + [G : C(a_1)] + \dots + [G : C(a_n)]) \Rightarrow p \mid |Z(G)|$. Since $Z(G) \leq G$, we know $|Z(G)| \mid |G|$. Therefore we have $p \mid |Z(G)| \mid p^k$ which implies $|Z(G)| = p^l$ for some positive integer $l \geq 1$. ■

b. proposition: If $|G| = p^2$ for some prime p , then $G \cong \mathbb{Z}_{p^2}$ or $G \cong \mathbb{Z}_p \times \mathbb{Z}_p$.

proof Consider the normal subgroup $Z(G)$ of G . By part (a) we know $|Z(G)| = p$ or p^2 . If $|Z(G)| = p^2$ then $Z(G) = G$ and G is Abelian. If $|Z(G)| = p$, then $|G/Z(G)| = p$ and is therefore cyclic. By theorem, if $G/Z(G)$ is cyclic then G is Abelian. In either case we have G is Abelian and the result follows from the Fundamental Theorem of Finite Abelian Groups. ■

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Problem 3:

Let G be a group of order 99. Prove that there exists a subgroup H of G of order 3 (this part should be a one-sentence proof), and a unique subgroup K of G such that $K/H \cong \mathbb{Z}_3$ and $G/K \cong \mathbb{Z}_{11}$.

proof By Sylow's first theorem, since $3||G|$ there exists a subgroup H of order 3. By Sylow's 2nd theorem we know this H must lie inside a Sylow 3-subgroup of G . By Sylow's 3rd theorem, we know the number of Sylow 3-subgroups must divide the order of G and be congruent to 1 mod 3. The divisors of G are 1, 3, 9, 11, 33, 99. The only one of these congruent to 1 mod 3 is 1. Thus, there is only 1 Sylow 3-subgroup, call it K . By a corollary to Sylow's 2nd theorem, this implies the K is normal in G . Furthermore $|K| = p^2$. Thus, we know K is Abelian, and therefore every subgroup of K is normal in K . Thus H is normal in K and we can consider the quotient group K/H . This group will have order $|K|/|H| = 9/3 = 3$, and is thus cyclic and isomorphic to \mathbb{Z}_3 . Similarly, since K is normal in G we can consider the quotient group G/K which will have prime order 11 and thus will be isomorphic to \mathbb{Z}_{11} .

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Problem EC:

Let G be the group of all $n \times n$ diagonal matrices with ± 1 in the diagonal entries. What is the isomorphism class of G ?

solution

$|G| = 2^n$, G is Abelian, and every element of G has order 2. Thus, by the Fundamental Theorem of Finite Abelian Groups $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$.

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