Problem 1:

a. **proposition:** If $G/Z(G)$ is cyclic then $G$ is Abelian.

**proof** If $G/Z(G)$ is cyclic then there exists some generator $gZ(G)$ such that $<gZ(G)> = G/Z(G)$. Let $a, b$ be arbitrary elements of $G$. $aZ(G) = (gZ(G))^i = g^iZ(G)$ for some $i$, and $bZ(G) = (gZ(G))^j = g^jZ(G)$ for some $j$. $aZ(G) = g^iZ(G)$ implies $a = g^iz_1$ for some $z_1 \in Z(G)$. Similarly, $bZ(G) = g^jZ(G)$ implies $b = g^jz_2$ for some $z_2 \in Z(G)$. Thus $ab = (g^iz_1)(g^jz_2) = g^i(g^jz_1z_2)$ since elements of the center commute with all elements of $G$. We have $ab = g^ig^jz_1z_2 = g^{i+j}z_1z_2 = g^ig^jz_1z_2 = g^ig^jz_2z_1 = g^jg^iz_1z_2 = ba$. Since $a$ and $b$ were arbitrary, we’ve shown the group operation on $G$ is commutative. Thus $G$ is Abelian.

**proposition:** If $N$ is a normal subgroup of $G$ and $N$ and $G/N$ are both $p$-groups, then $G$ is a $p$-group.

**proof** Let $x$ be an arbitrary element of $G$. Consider the coset $xN$ in $G/N$. If $G/N$ is a $p$-group, then $|xN| = p^k$ for some $k \in \mathbb{Z}^+$. In other words $(xN)^{p^k} = x^{p^k}N = N$. By properties of cosets, this implies $x^{p^k} \in N$. If $N$ is also a $p$-group, then every element of $N$ has order equal to some power of $p$. Thus $|x^{p^k}| = p^j$, i.e. $(x^{p^k})^{p^j} = x^{p^{k+j}} = e$. This implies $|x||p^{k+j}$, which implies $|x| = p^i$ for some positive integer $i$. Thus, since $x$ was arbitrary, the order of every element in $G$ is equal to $p$ to some power. By definition of $p$-group, this implies $G$ is a $p$-group. ■
Problem 2:
a. Use the class equation to prove that if \(|G| = p^k\) for some prime \(p\), then \(|Z(G)| = p^l\) for some integer \(l \geq 1\).

**Proof** The class equation states \(|G| = |Z(G)| + |cl(a_1)| + \ldots + |cl(a_n)|\) where the \(cl(a_i)\) are distinct conjugacy classes having more than one element (i.e. \(a_i \notin Z(G)\)). The class equation can also be written as \(|G| = |Z(G)| + |G : C(a_1)| + \ldots + |G : C(a_n)|\). For each \(a_i \in G\), \(|G : C(a_i)||G|\) Furthermore, if \(a_i \notin Z(G)\), \(|G : C(a_i)| > 1\). Thus, since \(|G| = p^k\), each term of the form \(|G : C(a_i)|\) in the class equation is divisible by \(p\). \(|G|\) is divisible by \(p\), thus \(p||Z(G)| + |G : C(a_1)| + \ldots + |G : C(a_n)||G|\). Since \(Z(G) \leq G\), we know \(|Z(G)||G|\). Therefore we have \(p||Z(G)||p^k\) which implies \(|Z(G)| = p^l\) for some positive integer \(l \geq 1\). ■

b. **Proposition**: If \(|G| = p^2\) for some prime \(p\), then \(G \cong \mathbb{Z}_{p^2}\) or \(G \cong \mathbb{Z}_p \times \mathbb{Z}_p\).

**Proof** Consider the normal subgroup \(Z(G)\) of \(G\). By part (a) we know \(|Z(G)| = p\) or \(p^2\). If \(|Z(G)| = p^2\) then \(Z(G) = G\) and \(G\) is Abelian. If \(|Z(G)| = p\), then \(|G/Z(G)| = p\) and is therefore cyclic. By theorem, if \(G/Z(G)\) is cyclic then \(G\) is Abelian. In either case we have \(G\) is Abelian and the result follows from the Fundamental Theorem of Finite Abelian Groups. ■
Problem 3:
Let $G$ be a group of order 99. Prove that there exists a subgroup $H$ of $G$ of order 3 (this part should be a one-sentence proof), and a unique subgroup $K$ of $G$ such that $K/H \cong \mathbb{Z}_3$ and $G/K \cong \mathbb{Z}_{11}$.

$\text{proof}$ By Sylow’s first theorem, since $3|\lvert G \rvert$ there exists a subgroup $H$ of order 3. By Sylow’s 2nd theorem we know this $H$ must lie inside a Sylow 3-subgroup of $G$. By Sylow’s 3rd theorem, we know the number of Sylow 3-subgroups must divide the order of $G$ and be congruent to 1 mod 3. The divisors of $G$ are 1, 3, 9, 11, 33, 99. The only one of these congruent to 1 mod 3 is 1. Thus, there is only 1 Sylow 3-subgroup, call it $K$. By a corollary to Sylow’s 2nd theorem, this implies the $K$ is normal in $G$. Furthermore $\lvert K \rvert = p^2$. Thus, we know $K$ is Abelian, and therefore every subgroup of $K$ is normal in $K$. Thus $H$ is normal in $K$ and we can consider the quotient group $K/H$. This group will have order $\lvert K \rvert/\lvert H \rvert = 9/3 = 3$, and is thus cyclic and isomorphic to $\mathbb{Z}_3$. Similarly, since $K$ is normal in $G$ we can consider the quotient group $G/K$ which will have prime order 11 and thus will be isomorphic to $\mathbb{Z}_{11}$.

$\blacksquare$
Show your work and make sure your answers are well organized, easy to follow, and properly explained.

**Problem EC:**
Let $G$ be the group of all $n \times n$ diagonal matrices with $\pm 1$ in the diagonal entries. What is the isomorphism class of $G$?

**solution**

$|G| = 2^n$, $G$ is Abelian, and every element of $G$ has order 2. Thus, by the Fundamental Theorem of Finite Abelian Groups $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$. 

Signature line: ______________________  ______________________