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Math 105

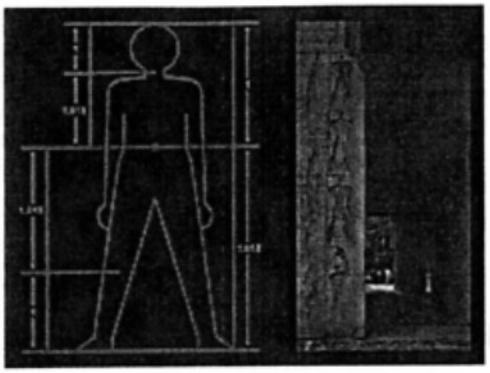
9 December 2003

Φ and Music

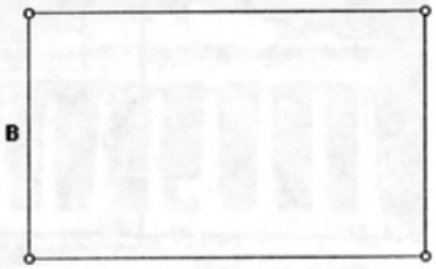
Through out western history Φ, also know and the Golden ratio or the divine proportion has had major effects on western cultural development and on western concepts of beauty, from architecture:

to art:

and to the understanding of science, nature, mathematics, and the structure of the human body:



The wide sweeping affects that the Golden Ratio has on western society is profound and is clearly illustrated by the Golden Rectangle:



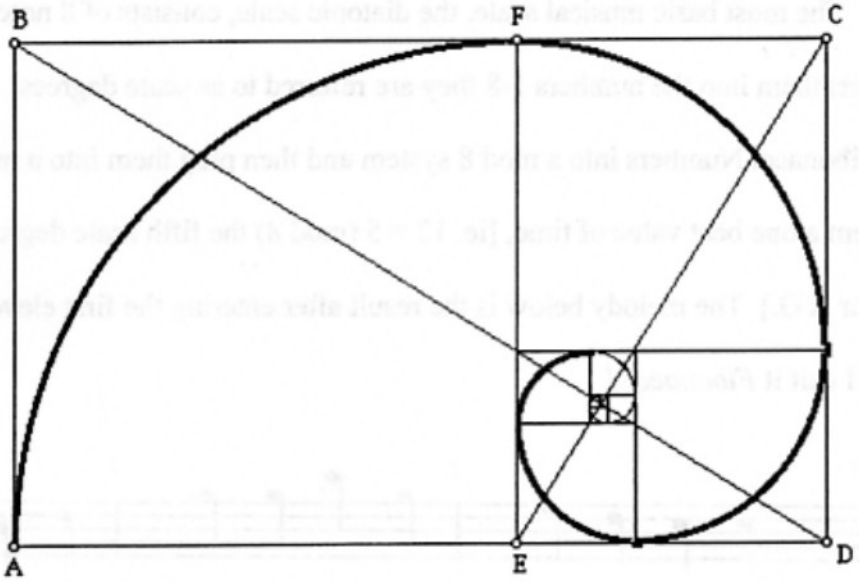
$$\frac{A}{B} = 1.618$$

The ratio of the long side over the short side of the golden rectangle equals Φ and the ratio of the short side over the long side equals the negative inverse of Φ. The golden rectangle is far more visually pleasing than a rectangle with a ratio that does not equal Φ; is not overlooked the modern consumer society:

Every one of the credit cards, driver's licenses, and phone cards that are carried in wallets through out the western world are perceived by the naked eye to be Golden Rectangles.

The golden ratio was discovered in ancient Greece and was perhaps their most revered number. Mathematically Φ (Phi) = $\frac{1+\sqrt{5}}{2}$ and equal approximately 1.61803…

If we take the equation $x^2 - x - 1 = 0$ and us the quadratic formula we find that Phi is one solution and that the negative inverse of Phi is the other. Mathematically Phi is also strongly linked with the Fibonacci numbers—an endless pattern of numbers where each number is equal to the sum of the previous two numbers (1,1,2,3,5,8,13…).



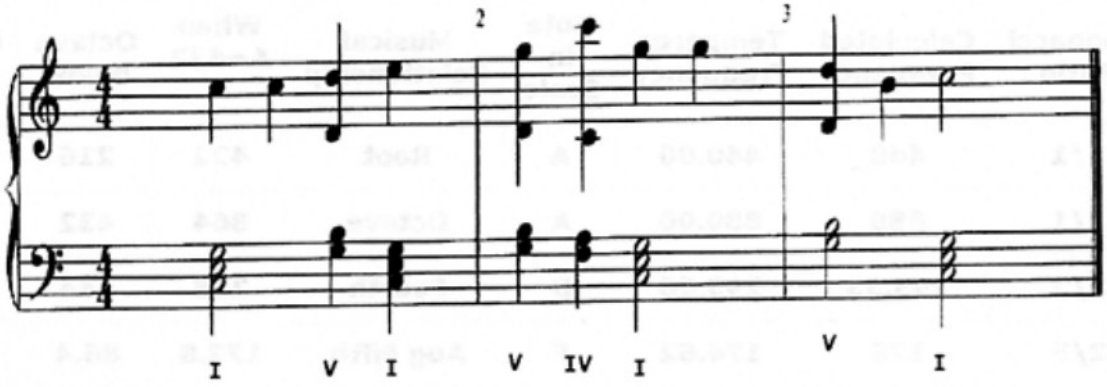
The above diagram is a Fibonacci Spiral. The lengths of the sides of the squares increase by following the Fibonacci sequence and creating the spiral. The point at which the blue lines intersect is the starting point of the spiral and the ratio of line BD over line CE is equal to the golden ratio. Furthermore as we increase the size of the spiral by adding more and more squares to the rectangle the ratio of the long side of the rectangle over the short side asymptotically approaches Phi and by the time the tenth square is added the

rectangle is "Almost Golden" meaning that to the naked the rectangle might as well be Golden.

Just as in architecture, nature and mathematics Phi and its relationship with the Fibonacci Numbers are prevalent in the structure of music. I choose to research the musical aspects of Phi because I have a strong previous knowledge of music. I decided that before doing research on the relationship between Phi and music I would attempt to find some relationships on my own and what I found astounded me. I decided that the best place to begin would be the Fibonacci Numbers. I made this decision because of the presence of the numbers 1, 3, 5, 8, and 13 all of which are important to the basic structure of music. The most basic musical scale, the diatonic scale, consists of 8 notes and when you convert them into the numbers 1-8 they are referred to as scale degrees. I decided to convert Fibonacci Numbers into a mod 8 system and then plug them into a musical song giving them a one beat value of time, [ie. 13 = 5 (mod 8) the fifth scale degree in the key of C Major is G.] The melody below is the result after entering the first eleven Fibonacci numbers, I call it *Fibonacci 1*

I found that after the eleventh number the this melody is repeated meaning that in Mod 8 the Fibonacci numbers repeat the eleven number pattern {1,1,2,3,5,8,5,5,2,7,1}. I was astonished to find that *Fibonacci 1* not only makes musical sense but also contains common and basic elements and patterns that appear in almost all western music, such as an arpeggio in notes 2-6, and something often referred to as a swell throughout the piece. However the most amazing thing I found in *Fibonacci 1* is in the final measure: the

second to last note, sometimes called the penultimate note, creates something called "voice leading" into the resolution on a C which in this key, the key of C, is the tonic. That probably sounds confusing to someone unfamiliar with music but it basically means that *Fibonacci 1* could not have ended with a better two notes. After this I added chords to the piece —it should be noted that a great number of chords could have been chosen but I chose chords that would best fit the piece and follow the most traditional guidelines. I call this song *Fibonacci 2*

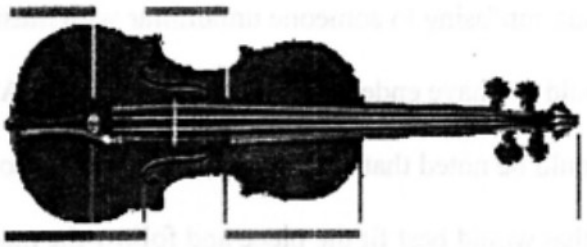


The only chords I needed to add to the song were the I (1) the V (5) and the IV (4). These chords are the three primary chords in music and by not having to resort to using other chords means that *Fibonacci 2* inherently follows traditional chord structures—the odds of this happening by selecting eleven random notes is astronomical.

I then put these discoveries aside and began to do research.

Fibonacci numbers appear almost constantly in western music. For instance a diatonic scale consists of 8 notes and leaves out 5 chromatic notes, a chromatic scale consist of 13 notes, and the most basic chord, a triad, consist of a 1 a 3 and a 5, are only a

few examples. Even the structure of some instruments follows patterns governed by the golden ratio:



Even the frequencies of many notes are related to Phi as demonstrated in this chart:

Fibonacci Ratio	Calculated Frequency	Tempered Frequency	Note in Scale	Musical Relationship	When A=432 *	Octave below	Octave above
1/1	440	440.00	A	Root	432	216	864
2/1	880	880.00	A	Octave	864	432	1728
2/3	293.33	293.66	D	Fourth	288	144	576
2/5	176	174.62	F	Aug Fifth	172.8	86.4	345.6
3/2	660	659.26	E	Fifth	648	324	1296
3/5	264	261.63	C	Minor Third	259.2	129.6	518.4
3/8	165	164.82	E	Fifth	162 (Φ)	81	324
5/2	1,100.00	1,108.72	C#	Third	1080	540	2160
5/3	733.33	740.00	F#	Sixth	720	360	1440
5/8	275	277.18	C#	Third	270	135	540
8/3	1,173.33	1,174.64	D	Fourth	1152	576	2304
8/5	704	698.46	F	Aug. Fifth	691.2	345.6	1382.4

Many of the important intervals are present on this chart such as the Root (1) the Fourth and Fifth. Also if you where to tune so that A= 432 (this is an antiquated practice) then the frequency of E natural is very close to Phi.

Also the musical climax of a piece often happens at the Phi point (point A is the Phi point if total time over time up to point A equals Phi), for example in a 32 measure song the climax is usually around the 20th measure. The climax of *Fibonacci 1* is on the 6th note of 11 notes making a ratio of 1.833 which is only about .215 of from Phi and with so short a song that is incredibly close.

I began to think that Phi was the basic influence behind the structure of Major keys, and chords, but what about minor keys. I was incredibly surprised to find that minor chords are also related to Phi. Thirteen is one of the Fibonacci Numbers and if you take a basic minor triad the note two octaves above the first note in the triad is the 13th note in the relative major scale.

Phi is an integral part of music composition but there are more aspects to music than was explored in this paper such as dynamics, rhythm, and timbre. In addition this paper only explored western music and many forms of eastern music don't follow the same patterns for instance the traditional music of India uses quarter steps instead of half steps and so their scales are profoundly different. It would be interesting to explore these other systems and aspects of music to find the influence of Phi but that would take a lifetime and a bit more paper than seven pages.