Appraising Commitments, Quasi-Commitments, and Guarantees

by

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Abstract

In this article, we price a hidden option embedded in the Oregonian Public Employee Retirement System (PERS). The Oregon PERS consists of two components; the Pension Program, which is a defined benefit plan, and an Individual Account Program, which is a defined contribution plan. In the defined contribution plan, members contribute 6 percent of their salary annually, which is matched by their employer and which goes into a larger fund invested and managed by the Oregon Investment Council (OIC). For members joining before 1996, the Oregon PERS guaranteed an annual rate of return of 8 percent. This guarantee created a hidden put option, which could be exercised whenever the annual rate of return on the PERS portfolio dropped below the guarantee. This put-option can be interpreted either as a European-style or an American style put-option. We price this hidden option using both the Black-Scholes model and the Binomial model. Our aim is demonstrating both the importance of recognizing government’s long-term liabilities and the difficulties of estimating them with precision. This demonstration has significance for governmental budgeting and financial reporting.

Keywords: options, present-value budgeting, public employee retirement systems, financial reporting, Oregon

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In spring of 2004, the Government Accounting Standards Board (GASB) issued Statements 43, Financial Reporting for Postemployment Benefit Plans other than Pension Plans, and 45, Accounting and Financial Reporting by Employers for Postemployment Benefit Plans other than Pension Plans. These standards require state and local governments to evaluate and disclose the cost of the postemployment benefits they have promised their employees. These include early retirement incentives, severance benefits, and health insurance premiums. Together with pensions, these benefits have grown twice as fast as wages and salaries over the past thirty years and now represent a substantial component of total compensation costs. Moreover, very small changes government’s commitments to its employees can often have large fiscal consequences. As Daniel Mullins and Michael Pagan have observed:

GASB’s requirement is a recognition of the potential significance of [an impending fiscal] bubble and the clear and present need for it to be addressed. The political expediency of generational cost shifting tends to insure a recurrent formation of such bubbles. ... A present and future challenge of financial management is to uncover and “prick” these bubbles while the financial implications are manageable.

This paper looks at the option values embedded in two of the most important postemployment-benefit choices made by the Oregon legislature in the recent past and suggests that it might well have acted differently, at least in the first instance, had it understood the real costs of its commitment. This shows the importance of evaluating and disclosing the cost of postemployment benefits that governments have promised their employees. Our example also demonstrates the difficulty of correctly appraising these costs and some of the reasons for these difficulties, the most important of which is the failure to specify precisely what government has promised or knowing how far its promises extend – when and if government will renege on its promises. At a minimum, we believe this uncertainty implies a need to supplement our ex ante estimates with ex post adjustments to government’s annual operating accounts.

Background

The utility of option pricing methods for valuing government commitments is, of course, no secret. Deborah Lucas, for example, outlined an approach similar to the one used here – Monte Carlo simulation of portfolio returns and a standard risk neutral options pricing technique, to estimate the cost to the United States government of guaranteeing a minimum return on investment income from private accounts financed by social security taxes. Others have used these techniques to estimate the cost of various govern-

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4 Deborah Lucas, Valuing Social Security Investment Guarantees. Department of Finance working paper, Northwestern University. Nov. 3, 2004. See also Deborah Lucas, Marvin Phaup, and Ravi Prasad, Esti-
ment insurance programs – everything from deposit and pension to crop and flood insurance. Moreover, even if one questioned the relevance of these methods to the full array of government commitments and quasi-commitments, making financial guarantees and mitigating financial risks remains a major function of government.

**Oregon’s Public Employees’ Retirement System**

Our saga begins about twenty-five years ago. Oregon’s retirement system (PERS) then had two components: a defined benefit plan, which was entirely publicly funded, and a defined contribution plan, which was not. Under the defined benefit plan, retirement benefits accruing to a member of PERS depended upon years of public service and salary. Under the second, benefits depended upon the individual’s contributions, which the state matched, and the performance of PERS’ investment portfolio. PERS’ members were required to contribute six percent of their earnings, but could elect to contribute more, up to a maximum of $5000.

Then, in the early 1980s, Oregon experienced an especially severe economic downturn, with three straight years of job losses. Faced with large revenue shortfalls, the state government responded with the usual expedients: deferring salary increases and scheduled maintenance, canceling plant and infrastructure investments, restructuring repayment schedules, imposing hiring freezes and restricting out-of-state travel, reducing transfers to local jurisdictions and school districts, and by borrowing funds from trust accounts. In this instance, much of the burden of retrenchment fell on public employees. The state froze wage and salary schedules – even merit and seniority increases were postponed – and borrowed heavily from PERS, primarily, but not entirely, by deferring contributions. To mitigate the effects of the salary freeze, which were worsened by inflation, the state assumed responsibility for the individual contribution to PERS, which increased members’ take-home incomes by six percent. To meet its new obligations to individual accounts, the state simply dipped further into the defined benefit portion of the PERS trust account.

Eventually prosperity returned – the state made good its obligations to PERS, restored merit, seniority, and cost of living adjustments. Nevertheless, the (state) courts refused to allow the legislature take back its commitment to fund the so-called individual contribution to PERS. At the time, that seemed a minor inconvenience. Thanks partly to the boom in stock and bond prices, both the defined benefit and the defined contribution components of PERS were fully funded – indeed, arguably substantially over funded. Moreover, the market value of PERS’ portfolio continued to increase faster than its actuarial obligations through the next economic downturn. Consequently, the state’s decision to withhold contributions to PERS’ defined benefit account in 1991 and 1992 went almost unnoticed.

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Instead, public and legislative attention focused on public employee salaries as the legislature once again balanced the budget at least in part by delaying wage and salary increases. Not surprisingly, the public employee unions pressed for compensation for the forced loans their members made to the state from current incomes and from the defined benefit portion of their retirement accounts.

And, once again, the legislature turned to the retirement account for redress, exchanging future benefits for current wage and retirement contributions. In this instance, the promise took the form of a guarantee of a return of at least 8 percent per annum on employees’ defined contribution accounts. Subsequently, members of the joint legislative budget committee have testified that they thought they were guaranteeing an average nominal return of 8 percent, which would have been a significant benefit, but not necessarily an unaffordable one given the average real returns of 7 to 9 percent that portfolios like PERS’s have earned historically.\(^6\)

This interpretation is not entirely consistent with the Department of Administrative Services’ (DAS) analysis of the legislation, which cited the 1986 stock market crash, when retirements practically stopped, in their assessment of the benefits of the proposal. DAS expressed concern that, failing such a measure, falling stock market values might reinforce the propensity to delay retirements at precisely the same time they would be of greatest value for budget balancing. Recessions often led the state to defer salary increases, which reduced the value of retirees’ defined benefits. As a consequence, elective retirements tended to drop during recessions, thereby reducing the potential salary savings that would result from holding positions vacant. Because the defined contribution portion dominated defined benefits in many individual retirement accounts, DAS felt that the proposed guarantee might more than offset this propensity. That would be the case, however, only if defined contribution accounts increased in real value. If this is what the analysts at DAS had in mind, they outsmarted themselves. Since investment performance is highly correlated with macroeconomic performance, this interpretation of the guarantee implicitly obligated the state to make very large contributions to PERS precisely when it could least afford them.

In any case, the public employees unions, officials of the Federal Retirement Trust, and the state courts embraced this more expansive interpretation of the guarantee. As the magnitude of potential liability became clearer to Oregon’s elected officials, they sought to rein it in. In 1996, the legislature restricted the guarantee to employees hired before that date (Tier 1 employees). Employees hired after 1996 (Tier 2 employees) were denied the guarantee.

Nevertheless, in 2001, the chickens came home to roost. At the end of 2000, Tier 1 employee contribution accounts held about $16 billion. Actual investment returns the following year were a loss of 7.17 percent. To satisfy a guarantee of an 8 percent increase, the state would have been obligated to contribute $2.4 billion to individual contribution accounts, plus a comparable amount for the state’s match, thus incurring liabilities of

$4.8 billion. The implicit obligation was approximately the same in 2002, for a total of approximately $9 billion – just about equal to the state’s general fund budget for the entire biennium.

Aghast at what they had wrought, the elected officials of the state presented the state’s Tier 1 employees with an ultimatum: use it or lose it. Employees who retired before the end of 2003 received full benefits. Tier 1 employees who elected not to retire by the end of 2004 or, in the case of teachers, the end of the 2004-5 school year, were deemed to have relinquished their guarantee. For a state employee with $600 thousand in her individual retirement account at the end of 2001, that meant the sacrifice of nearly $200 thousand.  

Subsequently, Tier 1 employees who chose not to retire sued the state for the contributions promised for 2001 and 2002. They have implicitly accepted the unilateral termination of the 8 percent guarantee and the loss of the present value of future benefits, however.

Calling all Options

Options give holders the right to buy or sell assets at specified prices and dates, without the obligation to do so (in other words, they allow holders to walk away from a transaction if they wish). The right to buy is generally dubbed a call, and to sell a put. There is also a difference between European-style options, which can be exercised (cashed in) only when they expire, and American-style ones, which can be exercised at any time during the option’s life. If, for example, a person held a European-style, six-month put option on a municipal bond, he could exercise the option only at the end of six months. If the bond price increased in the meantime, he would not exercise the option.

Figure 1 illustrates the spot-market value of a put option. Whenever the asset’s market price (S) is above its exercise or strike price (X), the put has no value. The option holder can get more by selling the asset in the market than by exercising the put. This is what the phrase, out-of-money, means. On the other hand, if the asset’s market price falls below the exercise price, the value of the put option will increase one-for-one with the decrease in the asset price, since the option holder can sell the asset at the exercise price even though its market price is less. This is what the phrase, in-the-money, means. The value of the put option will be the difference between the asset’s market price and the exercise price (X - S).

The holder of a put owns the right to sell the asset for the exercise price. The purchase price of an option is the option’s premium or, simply, its price. The profit from selling a put is the difference between its value and its premium. The purpose of this analysis is pricing the guarantee granted in 1994 and taken back ten years later. We presume that the PERS guarantee created a hidden put option, which could have been exercised whenever the rate of return on PERS earnings fell below 8 percent. Under the more expansive interpretation of the state guarantee, we assume that employees could exercise their options whenever it was advantageous to do so (in any period in which actual

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earnings were less than 8 percent). Under the less expansive interpretation, employees could exercise their options only at retirement (and would do so only if earnings averaged less than 8 percent per annum over all). Clearly, the more expansive interpretation of the guarantee makes it sound a lot like a series of American options and the less expansive interpretation like a European option. Nevertheless, we presume that, in both cases, it makes sense to recognize the option’s cost when it was granted (or taken) and not merely when or if it was exercised.

Pricing an option is a tricky affair, since it involves estimating the probability that the option will be worth exercising. In 1973, Fischer Black and Myron Scholes (with help from Robert Merton) published “The pricing of options and corporate liabilities.” This article reflected a fundamental insight – that all financial assets are essentially a mixture of forward contracts and options. By breaking assets down into their constituent parts, Black, Scholes, and Merton were able to produce what mathematicians call a closed-form solution to the problem of pricing a European option – an algorithm that churns out option prices by plugging in a set of values. These include the current price of the underlying asset, the option’s strike price (the price at which the purchaser can buy or sell something), the level of interest rates, the time to maturity, and the volatility of the asset.

The last is key – option-pricing models try to calculate the likelihood of an option being in the money, which depends on how much the underlying asset value is likely to fluctuate over its life. The more volatile the asset value, the more likely it is that the option will be exercised—and the more valuable the option. Volatility can be modeled with different distributions: normal, binomial, triangular, Weiner processes, etc. Black-Scholes relies on a standard normal distribution.

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The main alternative to Black-Scholes is what is the binomial pricing model, developed by John Cox, Stephen Ross and Mark Rubinstein. The binomial method is more complex than Black-Scholes, but if done properly, the price it produces can be more accurate, especially where American style options are concerned. We use both in the analysis that follows (see appendices).

**Methodology/Data**

As we have noted, the binomial model provides a good approximation for an American-style option and allows us to take account of the possibility that PERS members will exercise their options every period in which it would be advantageous to do so, reinvesting the product each period until mandatory termination of membership in the system at retirement. In contrast, the Black-Scholes model provides a good approximation for price of a European-style options and allows us to consider the possibility the PERS members can exercise their option only at retirement.

Therefore, we used both the binomial and the Black-Scholes model, to estimate the value of the PERS guarantee at its inception. Finally, we made the following simplifying assumptions about participation in the PERS defined contribution program prior to valuing the hidden put-option.

1. No salary inflation and no salary increases (That is, salaries do not grow over time, either nominally or in real terms). Hence, each year until retirement, PERS members will add a fixed amount to their retirement accounts (12 percent of $40 thousand in 1994 and of $60 thousand in 2001).
2. The typical PERS participant had been a member of the system for M years, assumed to be 11 in 1994 and 14 years when the guarantee was effectively terminated in 2001. We assume an average beginning portfolio value of $272 thousand in 1994 and of $782 thousand in 2001.
3. The typical PERS member expected to remain in the system N years to retirement, N = 15 years in 1994 and 11 years in 2001.

The maximum length of the option (time to maturity) reflects N-1 years, that is a PERS member adds his/her contribution to the beginning portfolio value after one year in the program and the option then runs until retirement (N-1 years later). For example if a member stays in the PERS-program for 10 years, the put option will run for 9 years. The underlying asset, S, will reflect the beginning balance, B, plus the annual contribution of 12 percent of salary, Q, since each contribution will go to the overall portfolio. The strike price, X, on the put option under the assumption about a guaranteed annual rate of return of 8 percent and an annual contribution of 12 percent of salary can be calculated using the following expression:

\[
X = \prod_{i=1}^{n} \left( (0.12 \cdot Q) \cdot (1.08)^{N-i} \right) + B(1.08)^{N-i} \quad \text{where } N = 1, 2, \ldots, N-1
\]

---

Since each PERS member makes an annual contribution to the portfolio, the overall put-option can be thought of as a portfolio of put options reflecting each annual contribution. Continuing the example from before, if the member stays in the PERS program for 10 years, the overall put position will consist of 9 put options. The first put option will be issued when the first contribution is made and will run for 9 years, the second put option will be issued when the second contribution is made and will run for 8 years, etc.

In order to find the present value of the overall put option, we discount each individual put option back to time 0 (1994 or 2001) using the risk free rate of return.

We based the risk free rate of return, \( r \), on the average yield of 30-year Treasury Bonds from January 1\(^{st} \) 1996 to December 31\(^{st} \) 2004, or \( r = 5.83 \) percent. We estimated the volatility of the underlying asset using the expected returns and volatility of the Russel 3000 stock index or the January 1\(^{st} \) 1996 to December 31\(^{st} \) 2004 period, \( \mu = 15 \) percent, \( \sigma = 17.14 \) percent. The Russel 3000 approximates the characteristics of the portfolio invested and managed by the Oregon Investment Council (OIC), i.e., slightly higher returns and risk than the market.

### Analysis

Using the methodology outlined in the previous section and varying salary levels, which changes the value of the underlying asset, \( S \), and the strike price, \( X \), and the time to maturity, the binomial model and the Black-Scholes produce dramatically different prices shown in Table 1 for put options:

<table>
<thead>
<tr>
<th>Salary ($)</th>
<th>Binomial</th>
<th>Black-Scholes</th>
</tr>
</thead>
<tbody>
<tr>
<td>$40,000</td>
<td>$762,102</td>
<td>$53,961</td>
</tr>
<tr>
<td>$60,000</td>
<td>$1,039,674</td>
<td>$95,990</td>
</tr>
</tbody>
</table>

Table 1 illustrates several basic points about the price of put options (which are supported by theory):

1. The price of a put option increases with annual salary levels (at a decreasing rate)
2. The price of a put option increases with time to maturity
3. The price of a series of American-style put options produced by the binomial model is far greater than the price of a single European-style put option estimated using Black-Scholes.

We have defined the price of the hidden put option discussed in this paper as the theoretical price individual members of Oregon PERS would be willing to pay for a guarantee of a minimum annual return on their contributions. What Table 1 also shows is that, taking the least expansive definition of the PERS commitment, a typical state employee
should have been willing to pay 125 percent of his or her annual salary for the guarantee.

\[
P (B = 272,000; Q = 40,000; N = 14; s = 17.14\%; r = 12\%, X = 802,000) = [54,000]
\]

Those figures can be interpreted to mean that, when the state guaranteed an 8 percent return on individual contributions, it made a one-time gift roughly equal to the general fund budget to its employees. When things got rough, it took the gift back. It seems unlikely to us that if the state’s elected officials had fully understood the magnitude of the gift they would have made this commitment.

Moreover, it is possible that the actual gift was far larger. This conclusion is consistent with values estimated using the binomial model depicted in Table 1. Taken seriously, these figures imply a gift of almost five times the state’s general fund budget for the biennium at its initiation and two-thirds that amount when the state finally clawed it back.

Conclusion

Budget formulation is concerned with decision making. Had Oregon’s elected officials better understood the consequences of their choices, it seems reasonable that they would have acted differently. At a minimum, it seems likely that they would have more clearly specified what it was they were promising state employees. This example rather dramatically demonstrates the utility of GASB’s general approach to evaluating and disclosing the cost of postemployment benefits.

There is also the matter of ex post reporting.\(^\text{10}\) GASB currently requires state and local governments to report the assets of defined benefit plans, obligations, and unfunded obligations (the actuarial difference between assets and obligations) for at least the last three years. Insofar as defined contributions plans are concerned, they are merely required to report benefits earned, recording a liability if benefits earned exceed the amount actually contributed to employee accounts. Before Standards 43 and 45, GASB made no provisions of any kind for the kind of option created by Oregon’s PERS and still has not detailed an approach to their appraisal or to their disclosure.

In contrast, the International Accounting Standards Board wants to expense these kinds of guarantees when they are awarded (in other words, the grant date). That approach would be consistent with the logic outlined here. It could also be reconciled with GASB standards governing defined benefit plans, in which unfunded liabilities are systematically amortized over the average remaining service life of active employees.

We would propose to go somewhat further. Under our approach, benefit plans would adjust the amounts expensed at grant dates in subsequent financial reports to account for subsequent events. In other words, entities would, when necessary, report extraordinary gains or losses to bring their accounts into line with the benefits actually exercised. We believe that this would reduce incentives on the part of governments and or plan

managers to manipulate appraisal values, which is, as we have noted, a tricky business in the best of circumstances, since divergences from estimated value would result in extraordinary charges. As The Economist (“So Many Options,” November 7, 2002) explained, “The issue boils down to this: is the granting of an option a once-only expense ..., the equivalent of paying the employee in cash? Or, is it a contingent liability, ... with the true cost becoming clear only when the option is either exercised or it expires? A once-only expense or a contingent liability: these are matters over which reasonable people can agree to differ.” We say the latter.

Discussion

The fundamental irony of public budgeting is that most fiscal commitments have consequences that reach beyond the temporal horizons of elected officials, not merely those that are concerned with distant, state-contingent outcomes. Financial economics’ standard answer to the problem of budgetary myopia is to treat every long-term commitment the way we have treated Oregon’s promise to its employees. This solution is usually called present-value budgeting.

This raises a second problem, however. The future is uncertain. How should we deal with this uncertainty in present-value budgeting? The orthodox economic approach to decisions involving future consequences calls for the analyst to measure a project’s expected net present value using a risk-adjusted discount rate. In contrast, the approach we have outlined here deals with uncertain, state-contingent benefits or costs by taking downstream options into account from the outset. This means assuming that a project’s execution will be suitably adjusted if and when new information becomes available – for example, finding robust solutions that will allow high value outcomes without regard to the state of nature that actually occurs or delaying commitments until better information is available. In any case, taking account of new information should lead to higher payoffs. In theory, we should be able to apply option-pricing methods to the valuation of non-financial or real options almost as easily as we can to financial commitments or guarantees.

11 In the United States, the Keynesian economics of the 1960’s had the effect of reinforcing the nearsightedness built into the legislative budget process by putting the federal executive budget on a cash (expenditure-receipts) basis. See the Report of the President’s Commission on Budget Concepts (Washington, DC: US Government Printing Office, October 1967).
14 Examples of the use of real options analysis in public administration are rare. Nevertheless, one very nice example is to be found in Daniel Carpenter’s analysis of the time needed for the United States Food and Drug Administration (FDA) to approve new drugs. He assumed the FDA knows the benefits of approval (expressed in terms of the drug’s efficacy, the size of the afflicted population, and the availability of substitutes, plus the political influence of potential users and the drug company submitting the application), but not its costs (expressed in terms the drug’s safety and, thereby, the threat error poses to FDA’s reputation). Carpenter assumed that safety is a standard Brownian motion or continuous-time Wiener process, a statistical distribution that exhibits intermittent exponential growth and decay, with an absorbing barrier at zero. Carpenter further assumed that time and repeated drug trials would produce better estimates of cost,
i.e., they will reduce the FDA’s uncertainty about a drug’s safety, which is to say the FDA would learn. Finally, the value Carpenter assigned to the FDA’s aversion to adverse drug reactions drove his model’s optimal stopping point. Implicitly, Carpenter concluded that the FDA should approve a new drug whenever the payoff to doing so equaled or exceeded the expected costs of approval plus the value to the FDA of further review. It is our impression that Carpenter outlines a typical administrative choice problem. Daniel Carpenter “Why Do Bureaucrats Delay? Lessons from a Stochastic Optimal Stopping Model of Agency Timing, with Applications to the FDA,” in Politics, Policy, and Organizations: Frontiers in the Study of Bureaucracy, George A. Krause and Kenneth J. Meier (editors), Ann Arbor: University of Michigan Press, 2003: 23-40.
Appendix 1: Binomial Model

The binomial model traces the evolution of an option’s underlying asset value via a binomial tree with a given number of branches (time periods) between the valuation date and the expiration. Each intersection of the binomial tree’s branches (node) represents a possible price for the underlying asset (from here on we refer to the underlying asset as the asset) at a particular time. This evolution of the price of the asset forms the basis for option valuation. The valuation process is iterative, starting at the end of each final node and then working backward through the tree to the initial node, which represents the valuation date. The result for the initial node is the option’s price.

The first step in this process is to identify the possible prices of the asset from the valuation date to the expiration date. Given the assumption that the asset price follows a multiplicative binomial process over discrete periods, the rate of return on the asset each period can have two possible values: \( u - 1 \) with probability \( p \), or \( d - 1 \) with probability \( p - 1 \). Thus, if the current asset price is \( S \), the asset price at the end of each period will be either \( uS \) or \( dS \). The factors \( u \) and \( d \) are referred to as the up and down factors. Figure 2 illustrates the evolution of the asset prices. We use the information about the volatility of the asset’s price and the number of periods to calculate up and down factors, according to the following expression:

\[
\begin{align*}
  u &= e^{\sigma \sqrt{t}} \\
  d &= e^{-\sigma \sqrt{t}} = \frac{1}{u}
\end{align*}
\]

Figure 2: The Evolution of the Asset Price (2-period model)

The price of the asset at the tree node \((j; i)\) is given by the following:

\[
S_{i,j} = S_0u^j d^{i-j}, \quad j = 0, 1, \ldots, i
\]
The second step of the process is to value the option at each of the expiration nodes. The
value of an option at the expiration date is its intrinsic value, which is the difference
between the exercise price, X and the asset's spot market price, S.

\[
\text{Call: } \max \left( S_0 u^j d^{i:j} X; 0 \right), j = 0,1,\ldots,N \\
\text{Put: } \max \left( X - S_0 u^j d^{i:j}; 0 \right), j = 0,1,\ldots,N
\]

The third step of the process is to calculate the option value of each earlier node using
the risk neutrality assumption. Under this assumption, today's fair price of a derivative
security is equal to the discounted expected value of its future payoffs. We calculate ex-
pected value using the option values from the last two nodes, weighted by their respec-
tive probabilities, i.e., the probabilities of either an up-movement \( p \) in the asset price or
a down-movement \( p - 1 \). We calculate today's fair price of the option by discounting
future expected values to present values using the risk free rate and the option's dura-
tion. Figure 3 illustrate this process of reverse engineering.

**Figure 3: Option valuation working backwards in the tree (2-period model)**

The probability of an up-movement in the asset price is given by the following expres-
sion:

\[
p = \frac{e^{rt} \Delta d}{u \Delta d}
\]

We calculate the price of an American option at each node using:

\[
\begin{align*}
\text{Call: } f_{i,j} &= \max \left( S_0 u^j d^{i:j} X; e^{rt} \left[ pf_{i+1,j+1} + (1 - p)f_{i+1,j} \right] \right) \\
\text{Put: } f_{i,j} &= \max \left( X - S_0 u^j d^{i:j}; e^{rt} \left[ pf_{i+1,j+1} + (1 - p)f_{i+1,j} \right] \right) \\
\text{Where } 0 \leq i \leq N & \text{ and } 0 \leq j \leq i
\end{align*}
\]
At each node it must be determined whether to exercise the option or keep it alive. In the analysis reported here, we calculated the overall value of the put option using the binomial model according to the following expression:

\[ P^{BN}(.12Q + B = S, t) = \sum_{i=1}^{n} PV(f_{i,j}) \]  

where \( n = 1,2,\ldots,N-1 \)

where \( f_{i,j} \) is the option value in the first node (at valuation time) of each binomial tree.

**Appendix 2: Black-Scholes model**

The Black-Scholes model presumes a continuous-time framework. This mathematical model is used to estimate the theoretical value of European call-and-put options over a finite time horizons.\(^{15}\) The key assumptions of the Black-Scholes model are:

- The price of the asset follows Geometric Brownian motion, with constant drift and volatility.
- The risk free interest rate is constant, and the same for all maturity dates.

The above assumptions lead to the following formula for the price of a European call option on a financial asset trading at a price, \( S \), where the option has an exercise price of \( X \), \( T \) a the cumulative normal distribution and \( T \) is years to maturity.

\[
C(S, t) = S \cdot N(d_1) - X e^{rT} \cdot N(d_2)
\]

\[
d_1 = \frac{\ln \frac{S}{X} + (r + \sigma^2/2)T}{\sigma \sqrt{T}}
\]

\[
d_2 = d_1 - \sigma \sqrt{T}
\]

Given put-call parity, the price of a European put option is:

\[
P(S, t) = X e^{rT} \cdot N(-d_2) - S N(-d_1)
\]

In the analysis reported here, we calculated the Black-Scholes model using the following expression:

\[
P^{BS}(.12Q + B = S, t) = \sum_{i=1}^{n} PV(X e^{rT} N(d_2) - S N(d_1))
\]

where \( n = 1,2,\ldots,N-1 \)

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\(^{15}\) Black, Scholes; *The Pricing of Options and Corporate Liabilities*; Journal of Political Economy 81, 1973
\[ d_1 = \frac{\ln \frac{S}{X} + \left( r + \frac{\sigma^2}{2} \right) t}{\sigma \sqrt{t}} \]

\[ d_2 = d_1 \sigma \sqrt{t} \]