# Time Value of Money

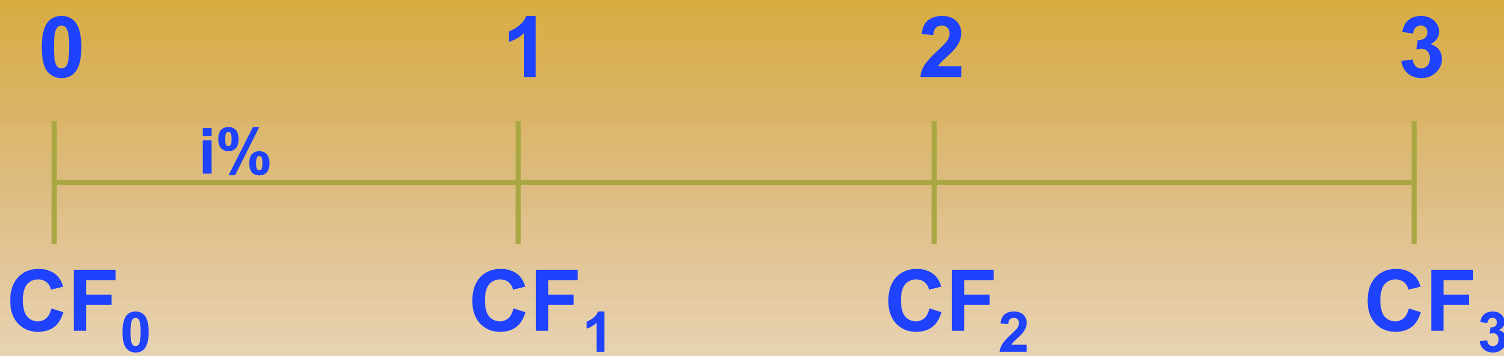
Future value

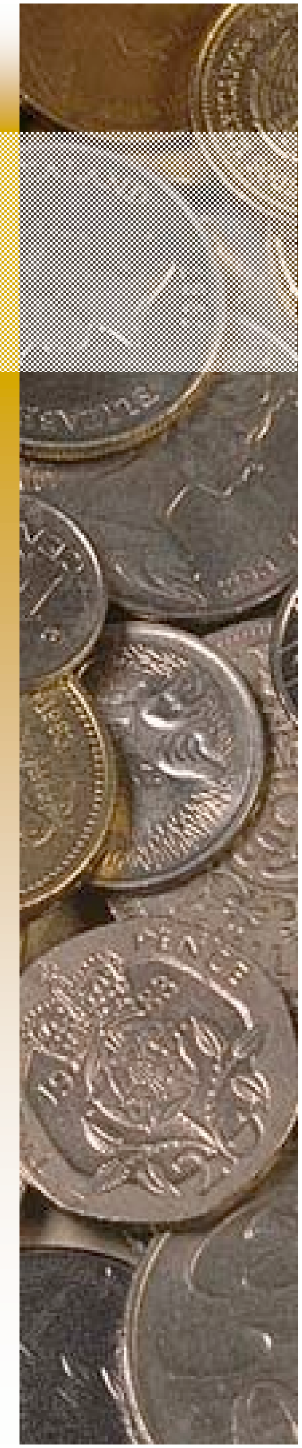
Present value

Rates of return

Amortization

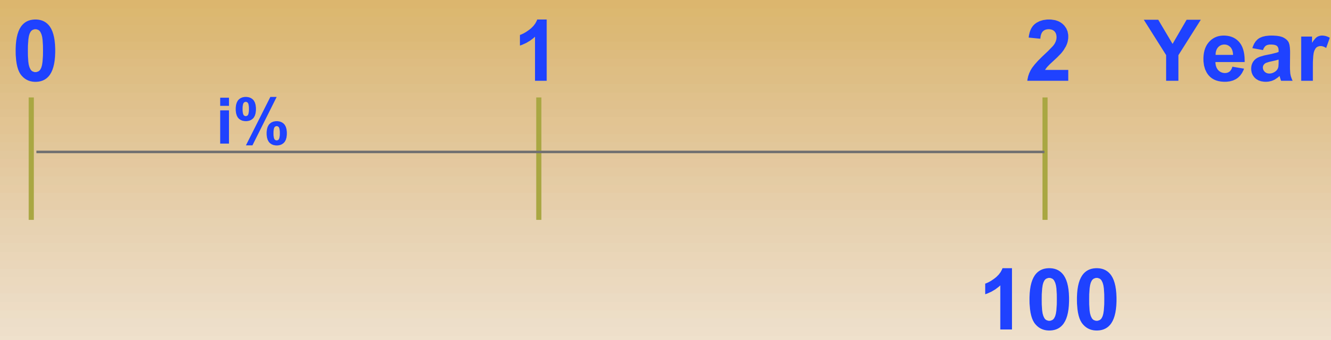
# Time lines show timing of cash flows.



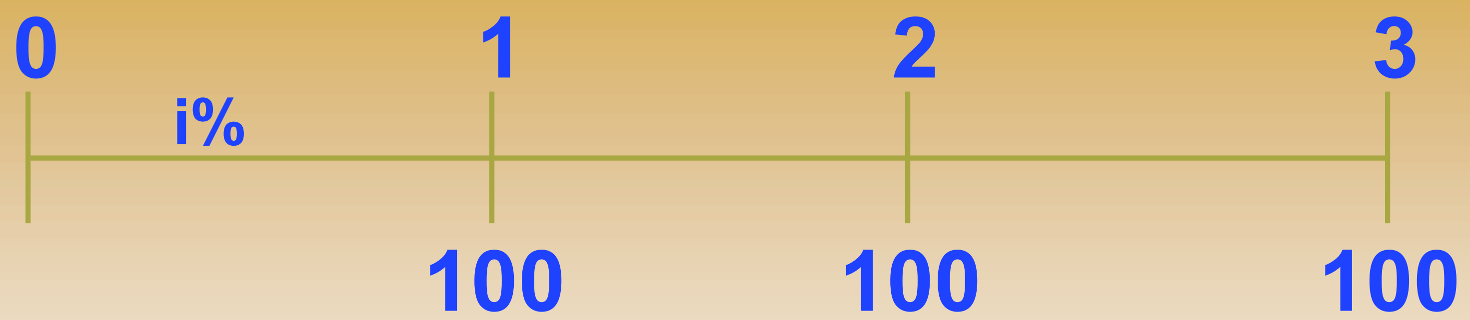



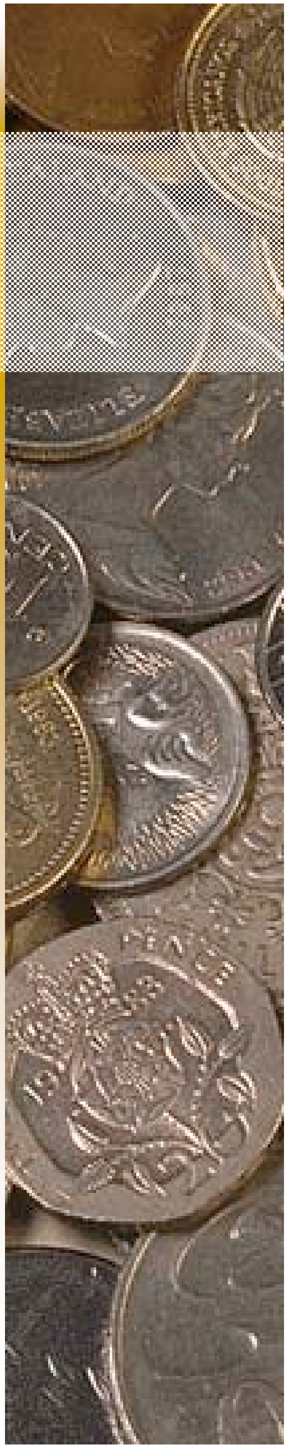
<u>Tick marks</u> at ends of periods, so Time 0 is today; Time 1 is the end of Period 1; or the beginning of Period 2.

# Time line for a $100 lump sum due at the end of Year 2.



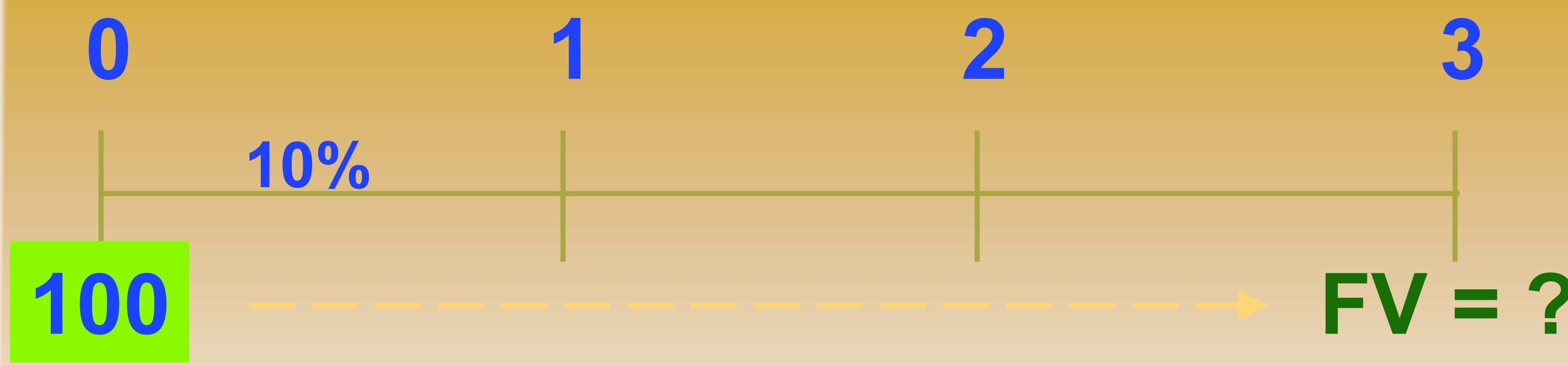
# Time line for an ordinary annuity of $100 for 3 years.



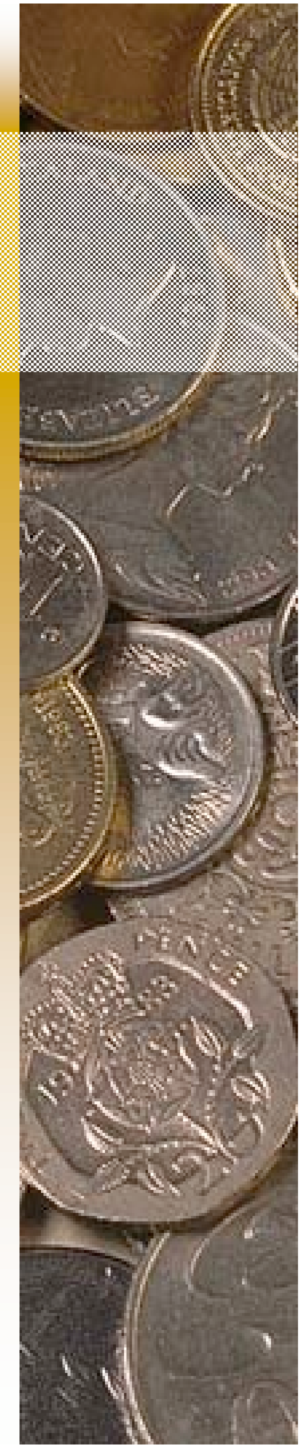



Time line for uneven CFs: -$50 at t = 0 and $100, $75, and $50 at the end of Years 1 through 3.

# What's the FV of an initial $100 after 3 years if i = 10%?

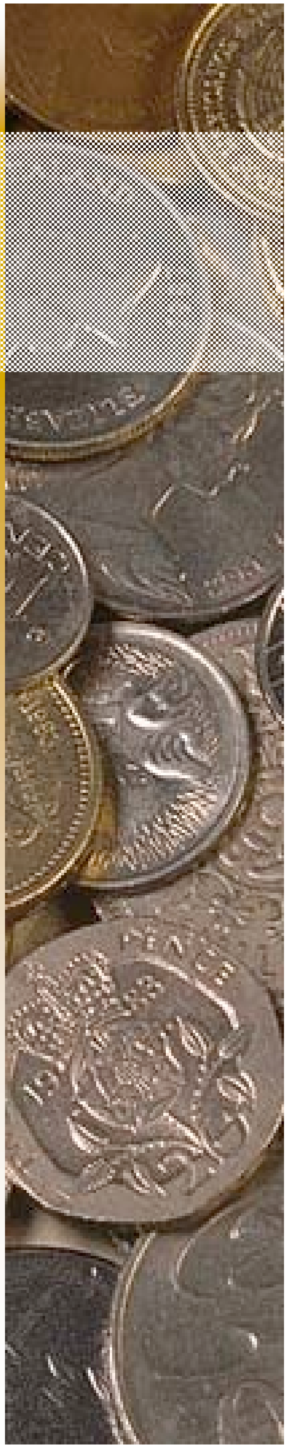



Finding FVs (moving to the right on a time line) is called **compounding.**



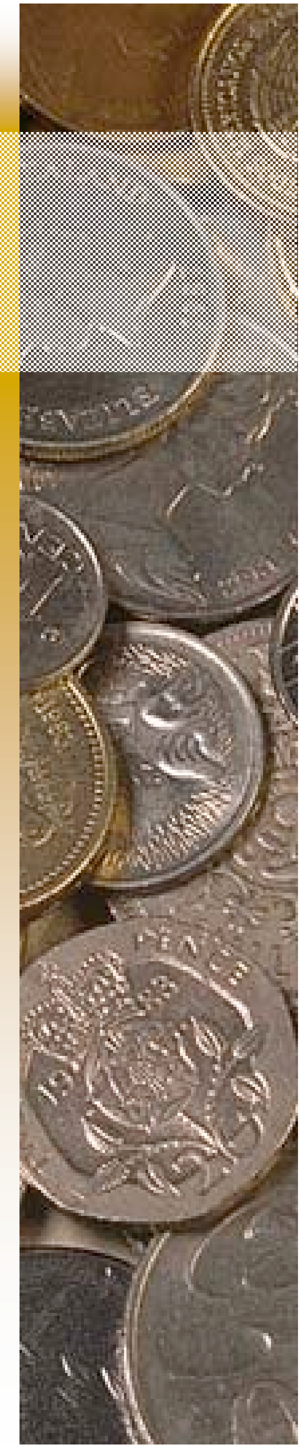
# After 1 year:

$$
\begin{aligned}
FV_1 &= PV + INT_1 = PV + PV\,(i) \\
&= PV(1 + i) \\
&= \$100(1.10) \\
&= \$110.00.
\end{aligned}
$$



# After 2 years:

$$FV_2 = FV_1(1+i) = PV(1 + i)(1+i)$$
$$= PV(1+i)^2$$
$$= \$100(1.10)^2$$
$$= \$121.00.$$

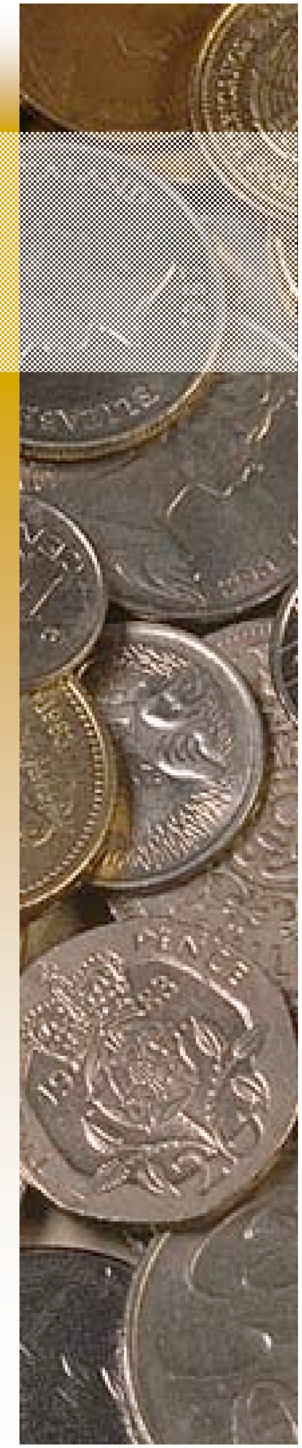


# After 3 years:

$$FV_3 = FV2(1+i) = PV(1+i)^2(1+i)$$
$$= PV(1+i)^3$$
$$= \$100(1.10)^3$$
$$= \$133.10.$$

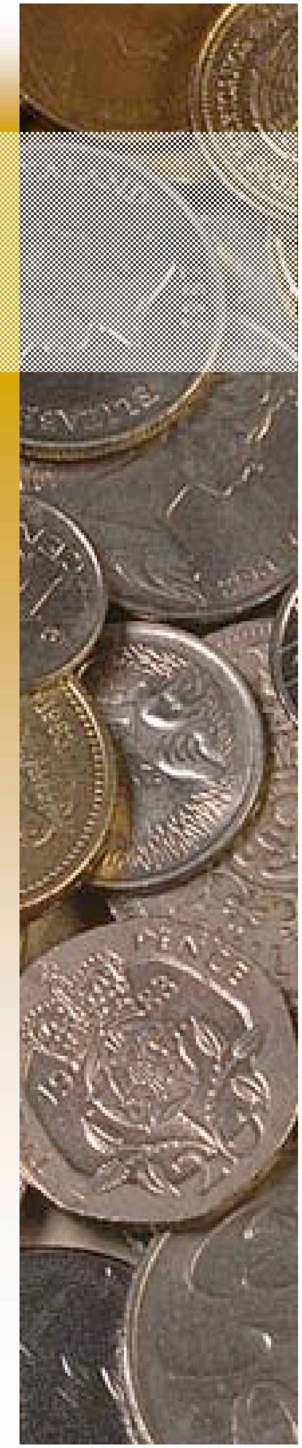
In general,

$$FV_n = PV(1+i)^n.$$

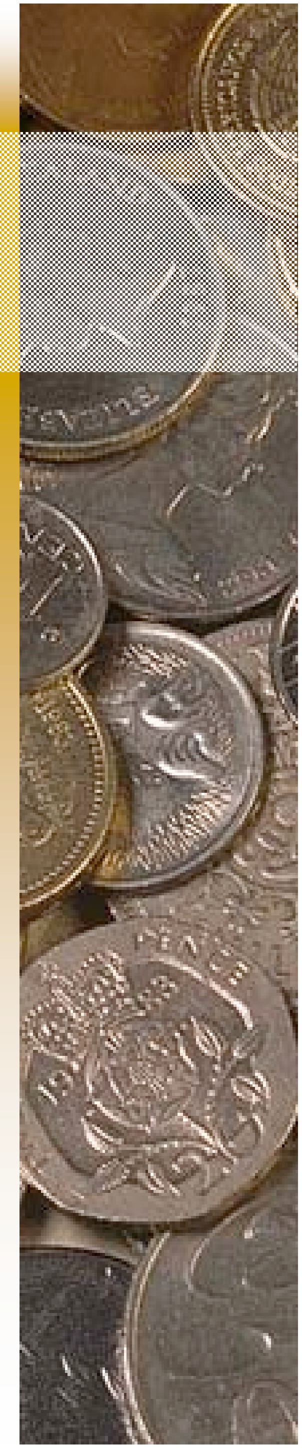


# Three Ways to Find FVs

- Solve the equation with a regular calculator (or use FV tables from your accounting text).
- Use a financial calculator.
- Use a spreadsheet.



# Financial Calculator Solution



Financial calculators solve this equation:

$$FV_n + PV(1+i)^n = 0.$$

There are 4 variables. If 3 are known, the calculator will solve for the 4th.

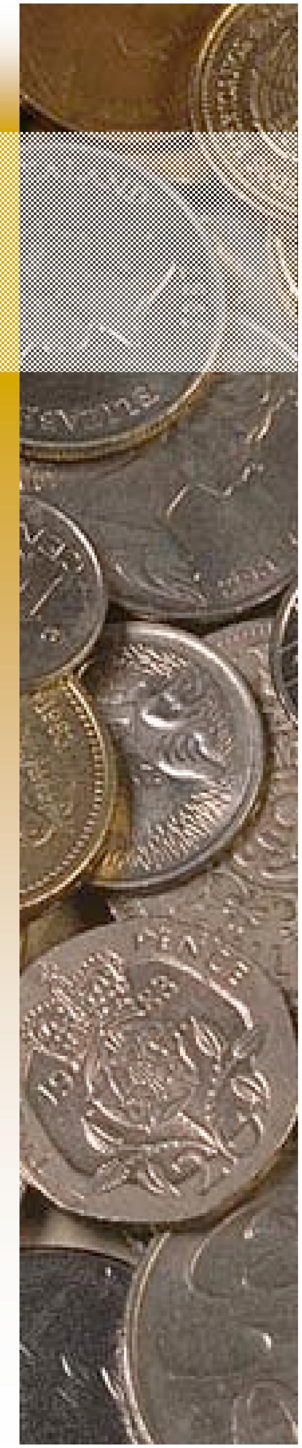
# Spreadsheet Solution

- **Use the FV function:**

**= FV(Rate, Nper, Pmt, PV)**

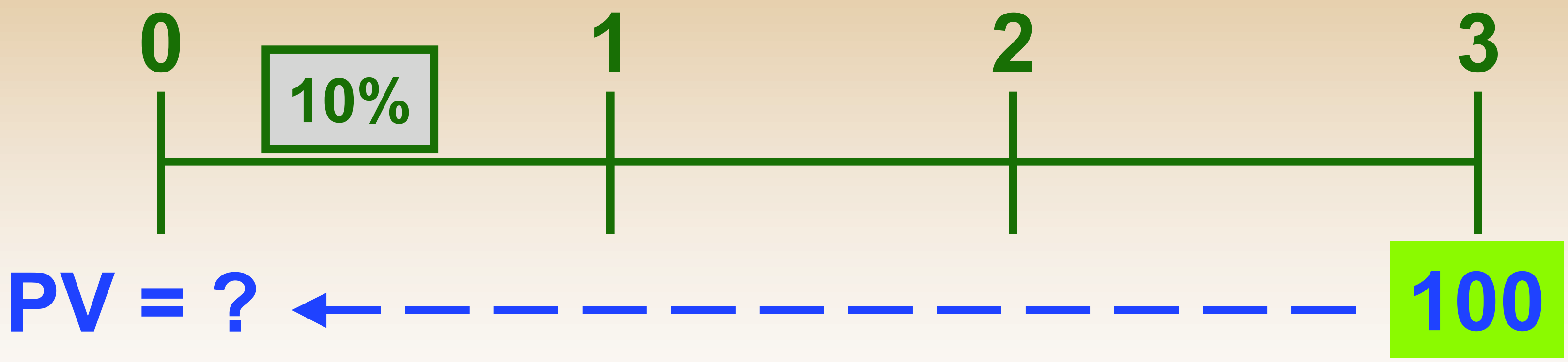
**= FV(0.10, 3, 0, -100) = 133.10**

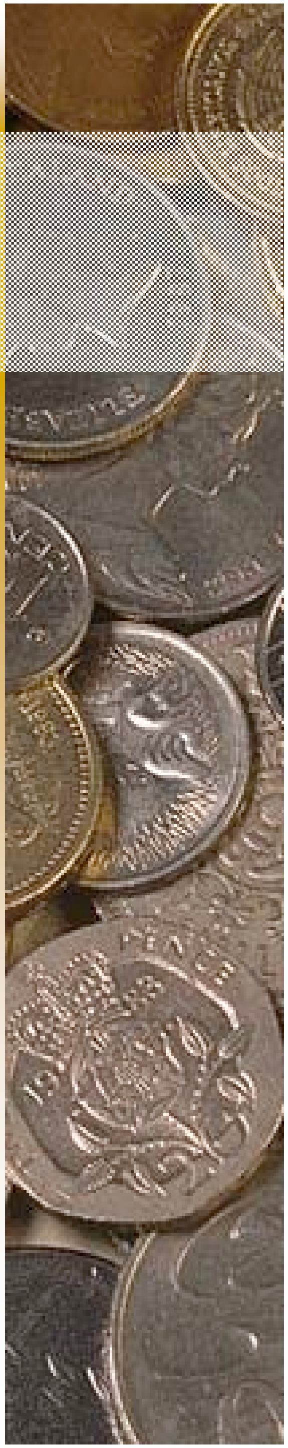
*see spreadsheet in Ch 02 Mini Case.xls.*



# What's the PV of $100 due in 3 years if i = 10%?

Finding PVs is discounting, and it's the reverse of compounding.



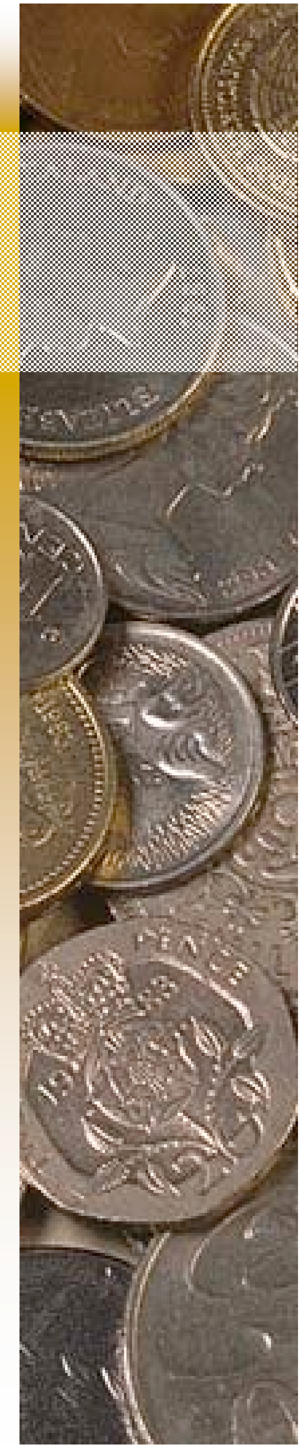



# Solve $FV_n = PV(1 + i)^n$ for PV:

$$PV = \frac{FV_n}{(1+i)^n} = FV_n\left(\frac{1}{1+i}\right)^n$$

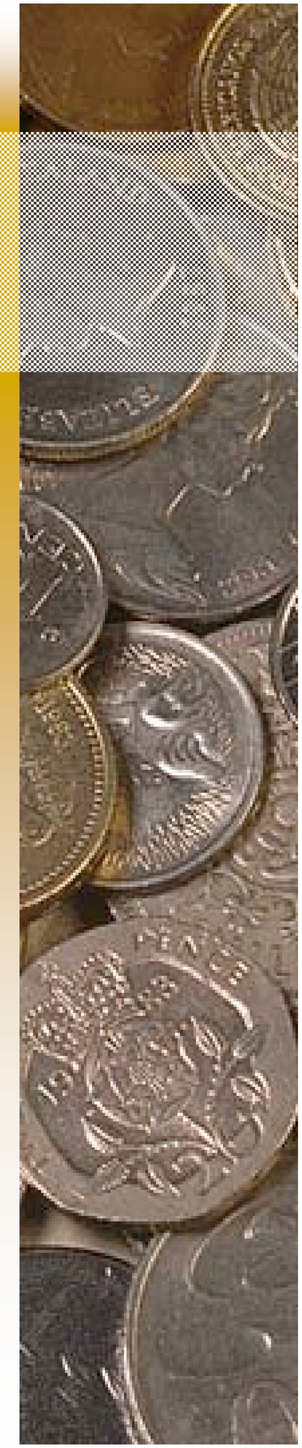
$$PV = \$100\left(\frac{1}{1.10}\right)^3$$

$$= \$100(0.7513) = \$75.13.$$

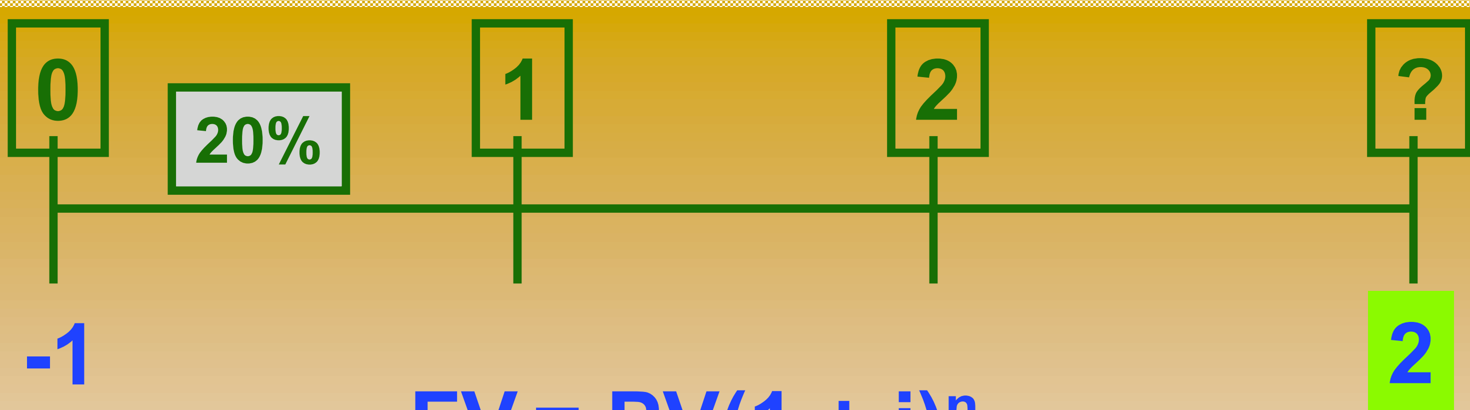


# Spreadsheet Solution

- **Use the PV function: see spreadsheet.**
  - **= PV(Rate, Nper, Pmt, FV)**
  - **= PV(0.10, 3, 0, 100) = 75.13**



# Finding the Time to Double




$$FV = PV(1 + i)^n$$

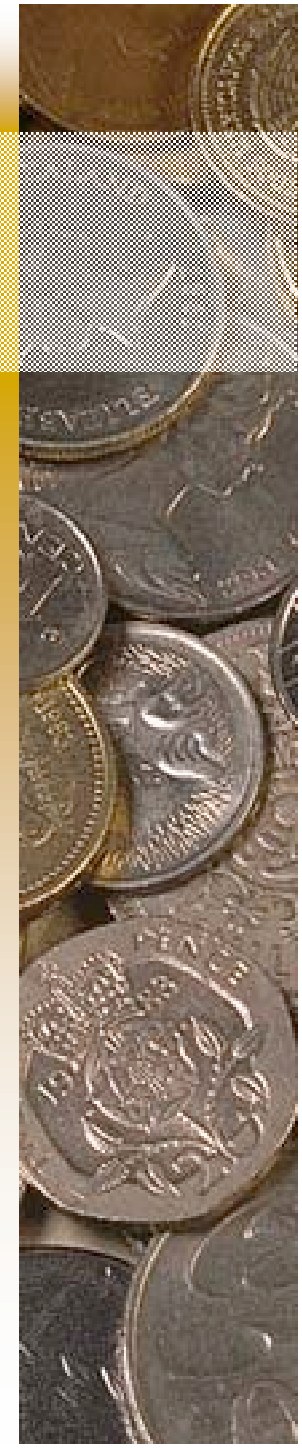
$$\$2 = \$1(1 + 0.20)^n$$

$$(1.2)^n = \$2/\$1 = 2$$

$$n LN(1.2) = LN(2)$$

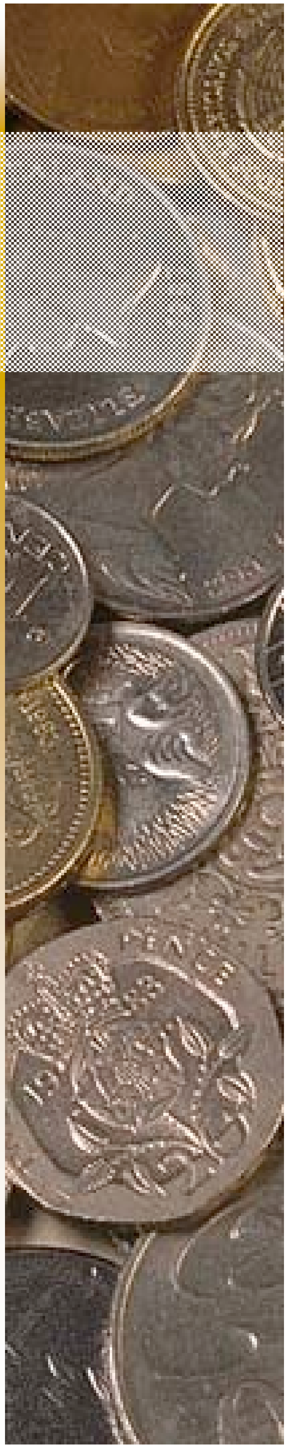
$$n = LN(2)/LN(1.2)$$

$$n = 0.693/0.182 = 3.8.$$

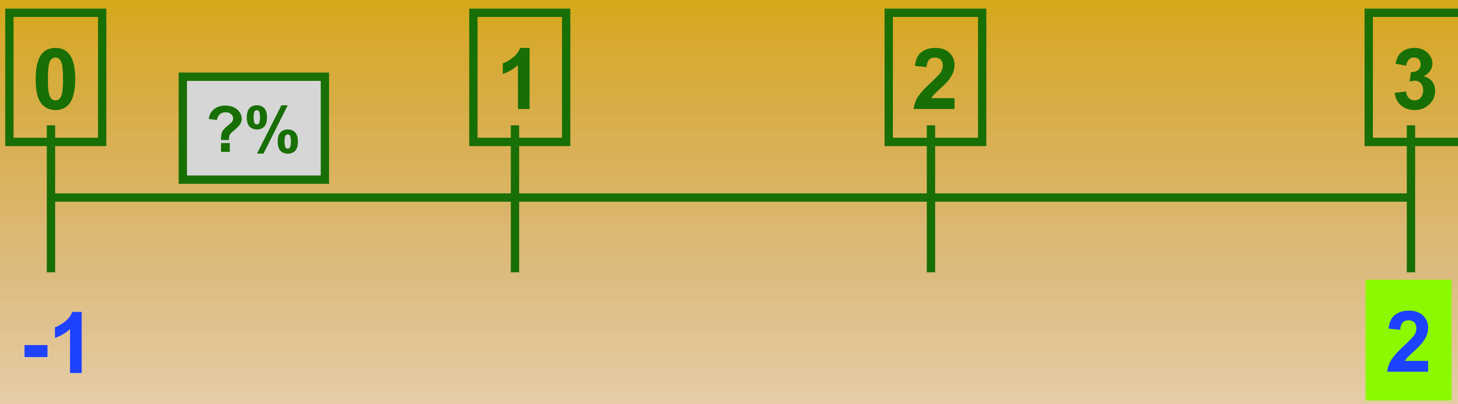


# Spreadsheet Solution

- Use the NPER function: see spreadsheet.
  = NPER(Rate, Pmt, PV, FV)
  = NPER(0.10, 0, -1, 2) = 3.8



# Finding the Interest Rate



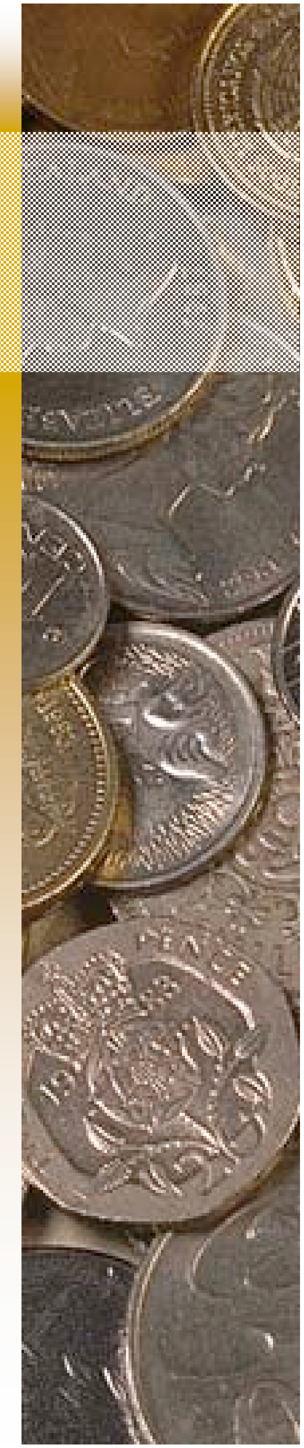

$$FV = PV(1 + i)^n$$

$$\$2 = \$1(1 + i)^3$$

$$(2)^{(1/3)} = (1 + i)$$

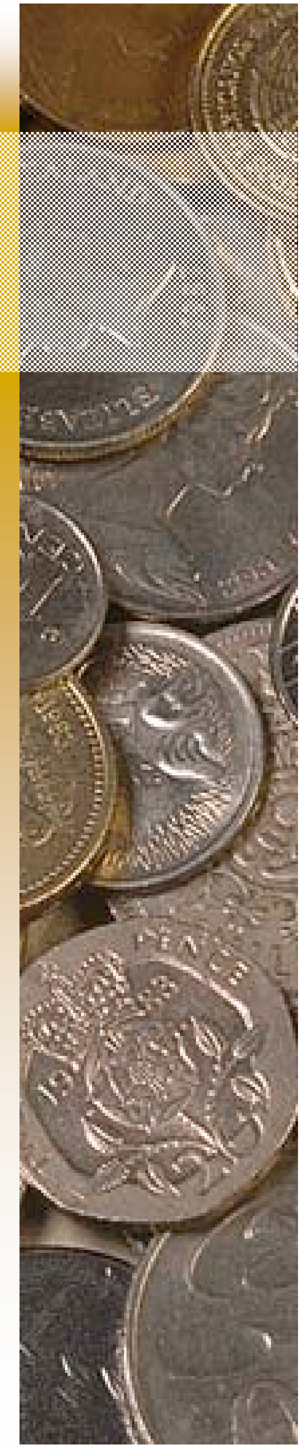
$$1.2599 = (1 + i)$$

$$i = 0.2599 = 25.99\%.$$

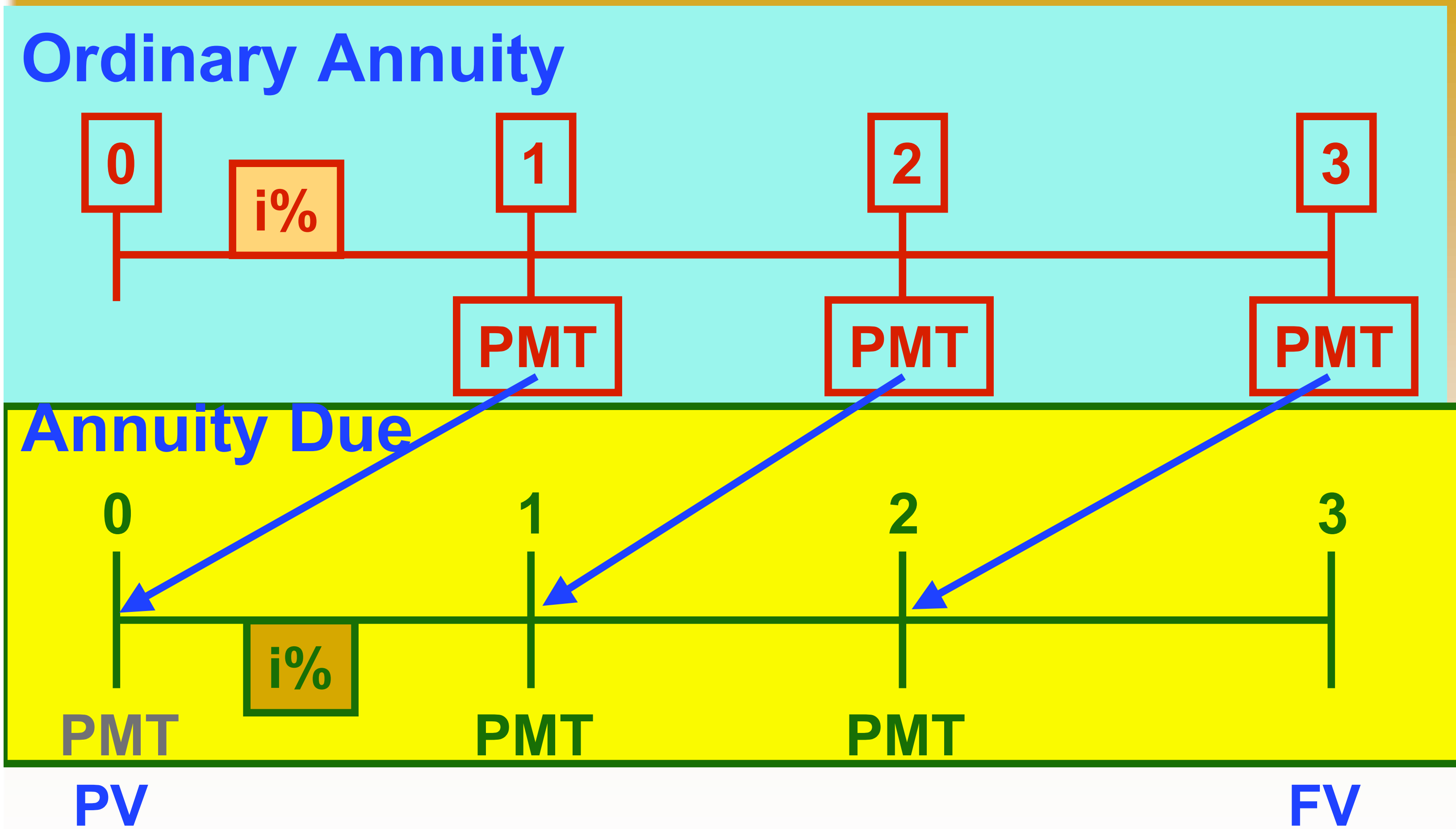


# Spreadsheet Solution

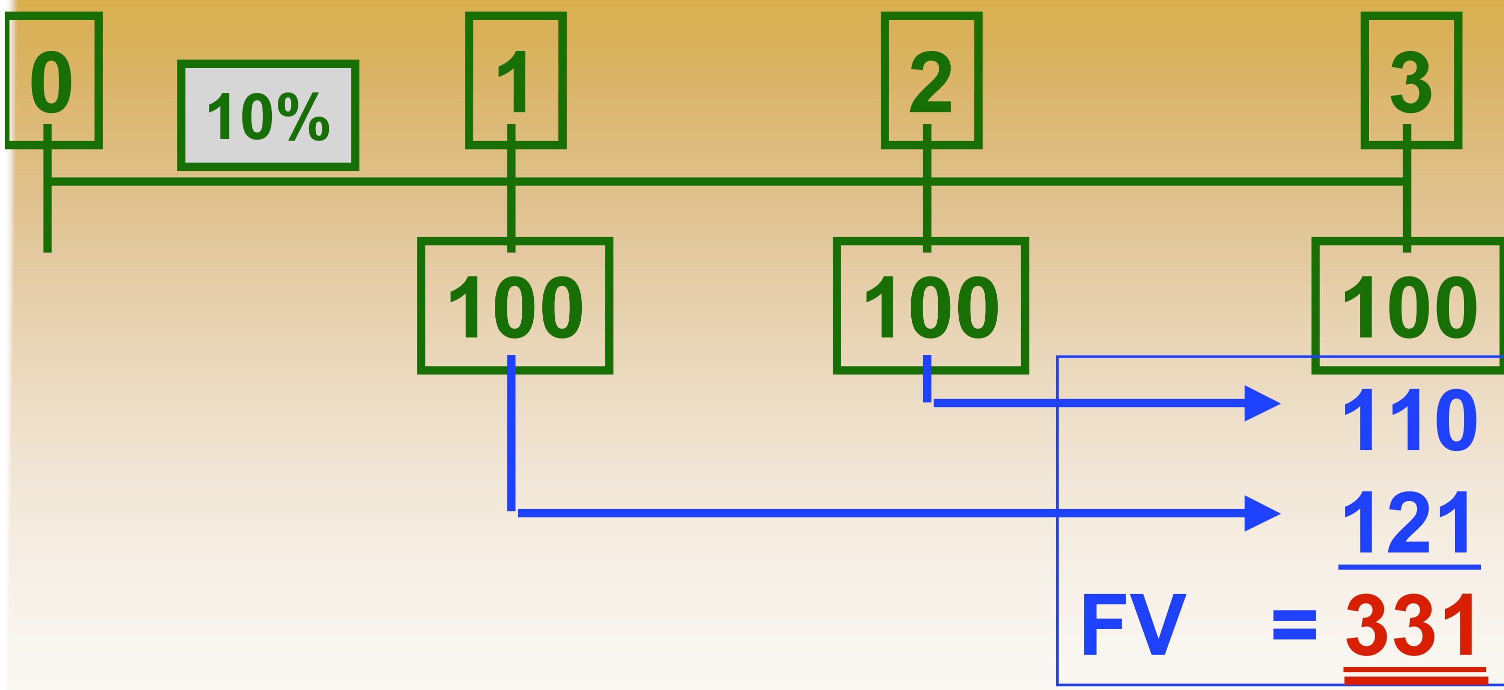
- Use the RATE function:
  - = RATE(Nper, Pmt, PV, FV)
  - = RATE(3, 0, -1, 2) = 0.2599

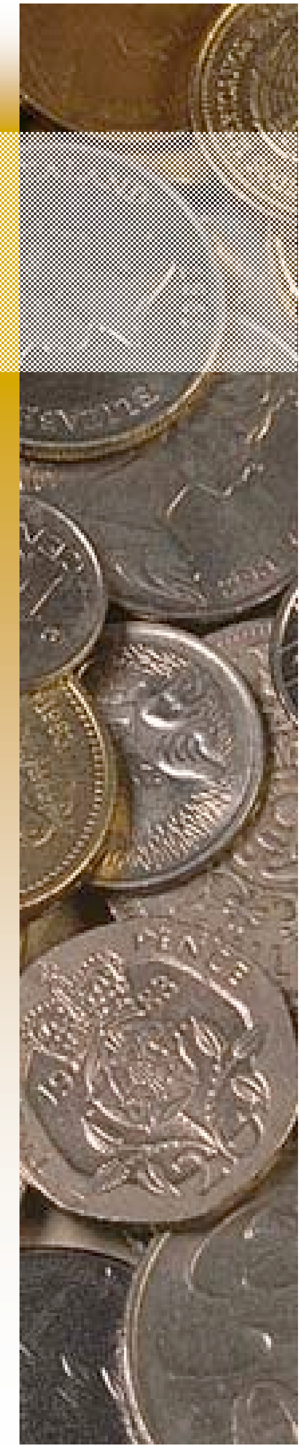


# What's the difference between an ordinary annuity and an annuity due?



# What's the FV of a 3-year ordinary annuity of $100 at 10%?



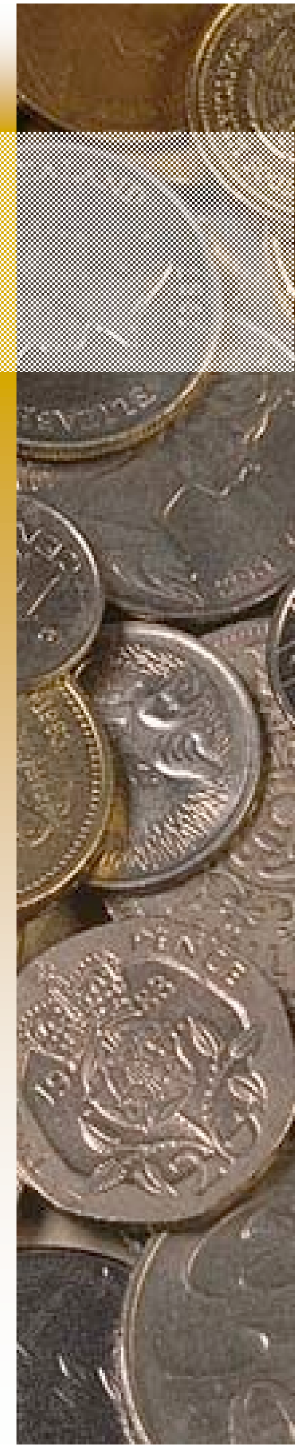



# FV Annuity Formula

- The future value of an annuity with n periods and an interest rate of i can be found with the following formula:

$$= PMT \frac{(1+i)^n - 1}{i}$$

$$= 100 \frac{(1+0.10)^3 - 1}{0.10} = 331.$$

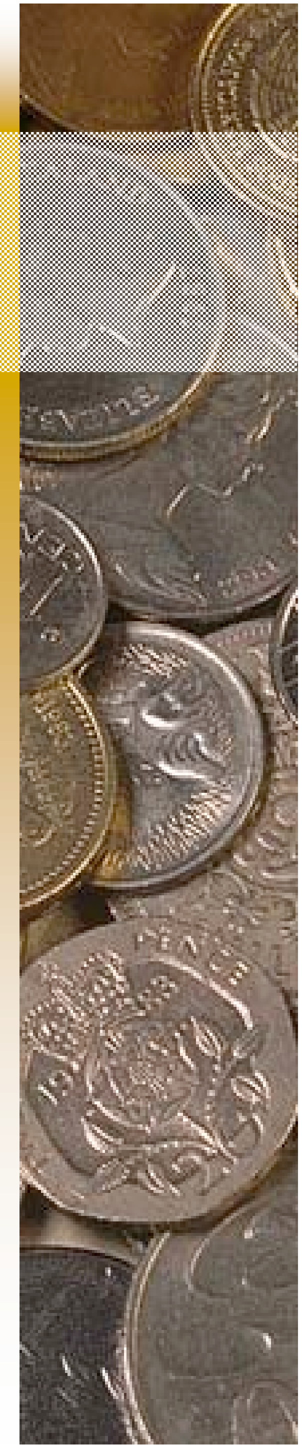


## Financial Calculator Formula for Annuities

Financial calculators solve this equation:

$$FV_n + PV\left(1+i\right)^n + PMT\frac{(1+i)^n - 1}{i} = 0.$$

There are 5 variables. If 4 are known, the calculator will solve for the 5th.

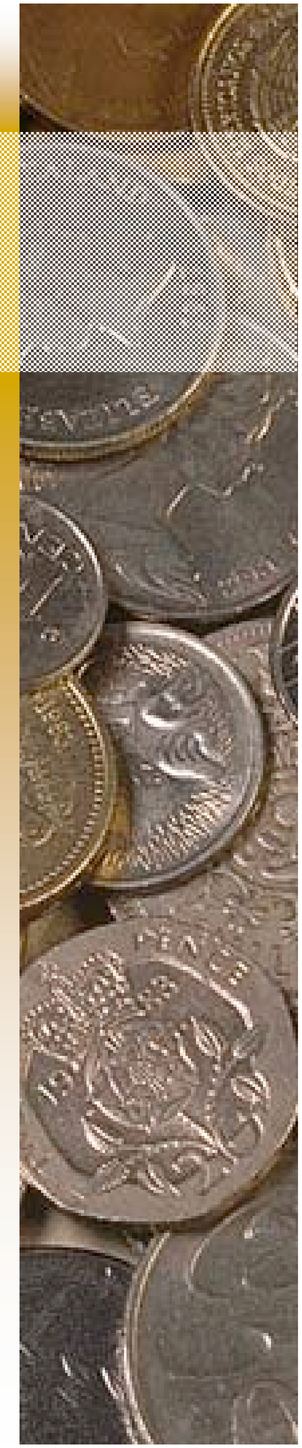


# Spreadsheet Solution

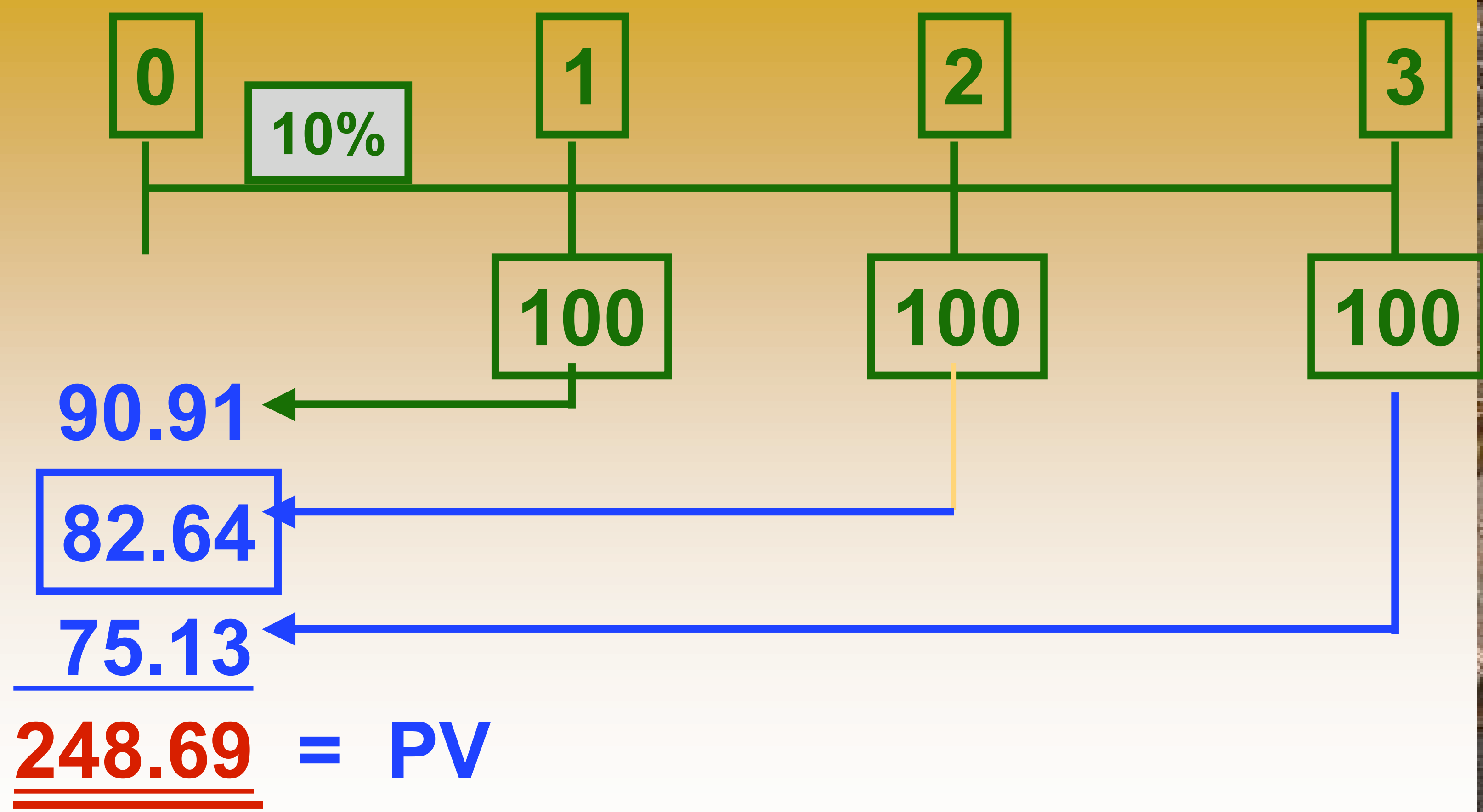
- Use the FV function: see spreadsheet.

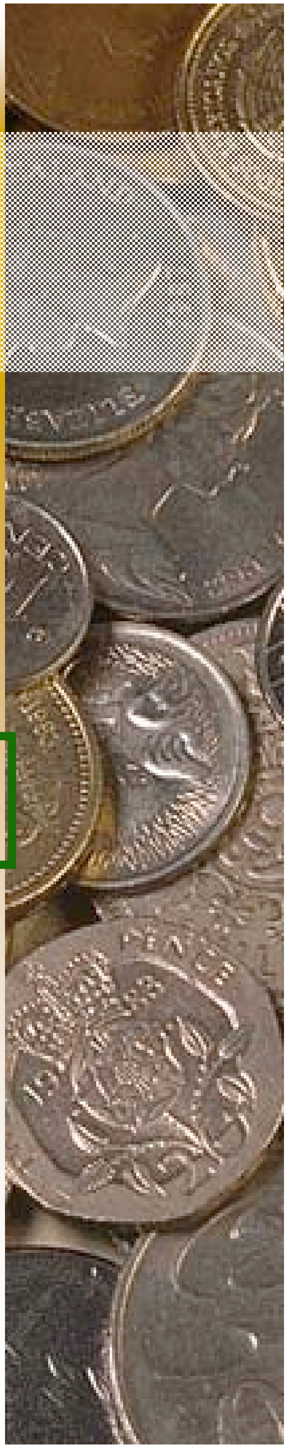
  = FV(Rate, Nper, Pmt, Pv)

  = FV(0.10, 3, -100, 0) = 331.00



# What's the PV of this ordinary annuity?



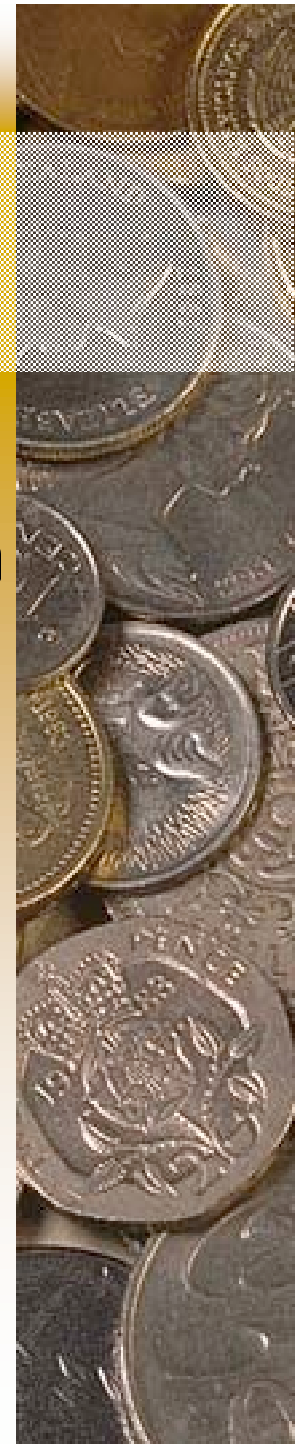



# PV Annuity Formula

- The present value of an annuity with n periods and an interest rate of i can be found with the following formula:

$$= PMT \frac{1 - \frac{1}{(1+i)^n}}{i}$$

$$= 100 \frac{1 - \frac{1}{(1+0.10)^3}}{0.10} = 248.69$$

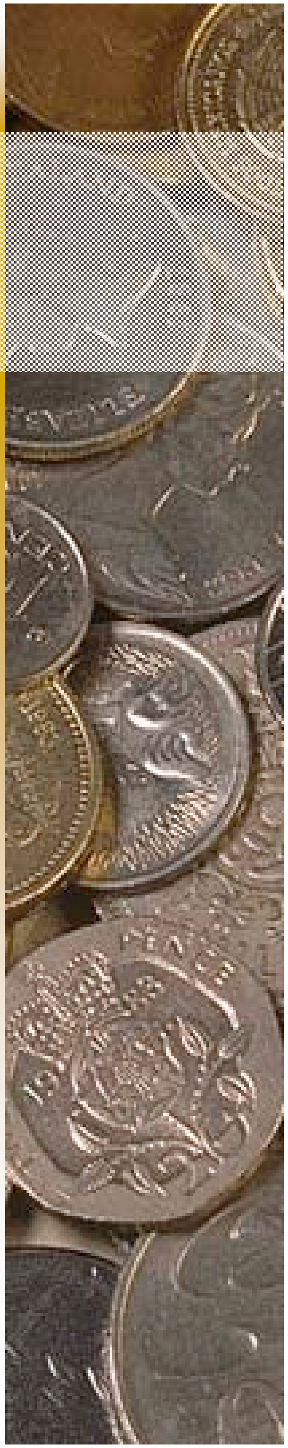


# Spreadsheet Solution

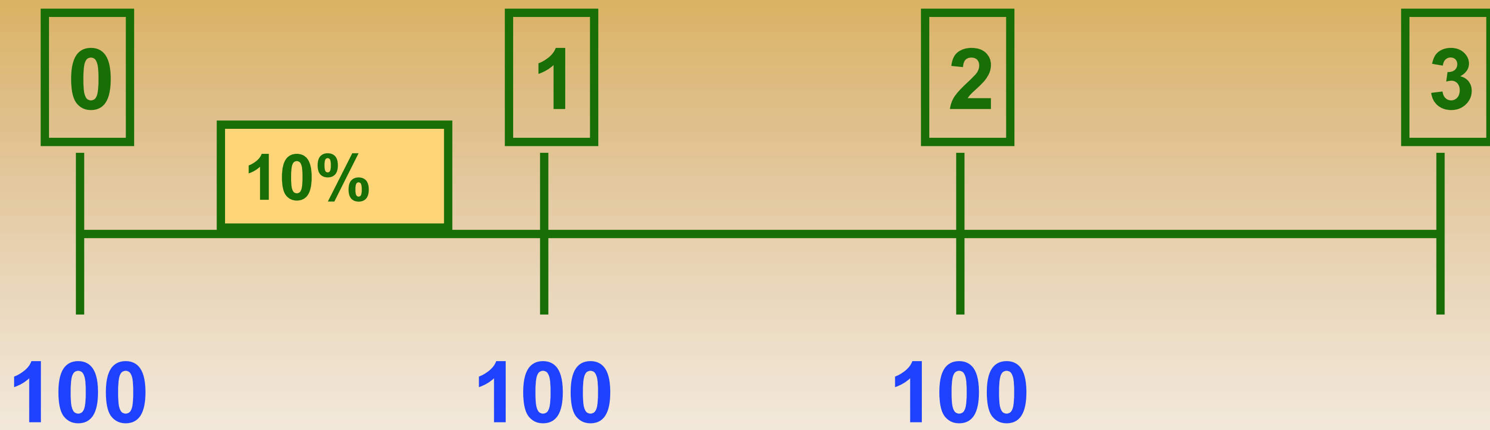
- Use the PV function: see spreadsheet.

  = PV(Rate, Nper, Pmt, Fv)

  = PV(0.10, 3, 100, 0) = 248.69

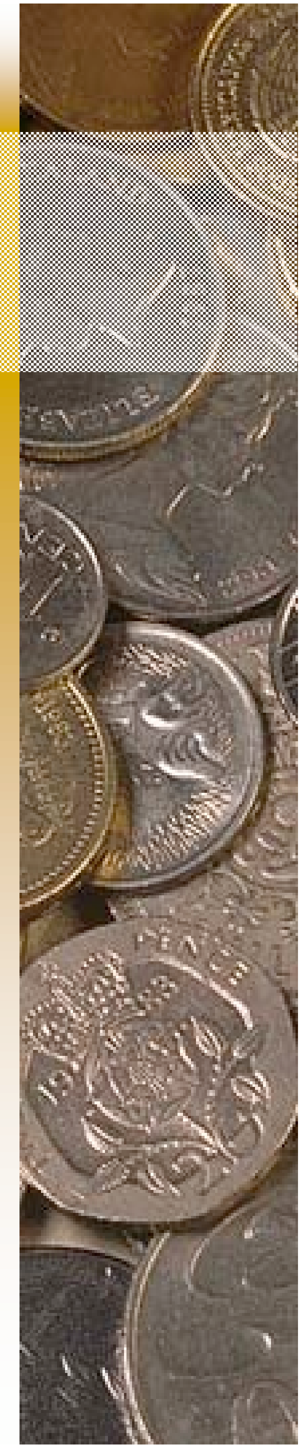


# Find the FV and PV if the annuity were an annuity due.



# PV and FV of Annuity Due vs. Ordinary Annuity

- PV of annuity due:
  - = (PV of ordinary annuity) (1+i)
  - = (248.69) (1+ 0.10) = 273.56
- FV of annuity due:
  - = (FV of ordinary annuity) (1+i)
  - = (331.00) (1+ 0.10) = 364.1



# Excel Function for Annuities Due

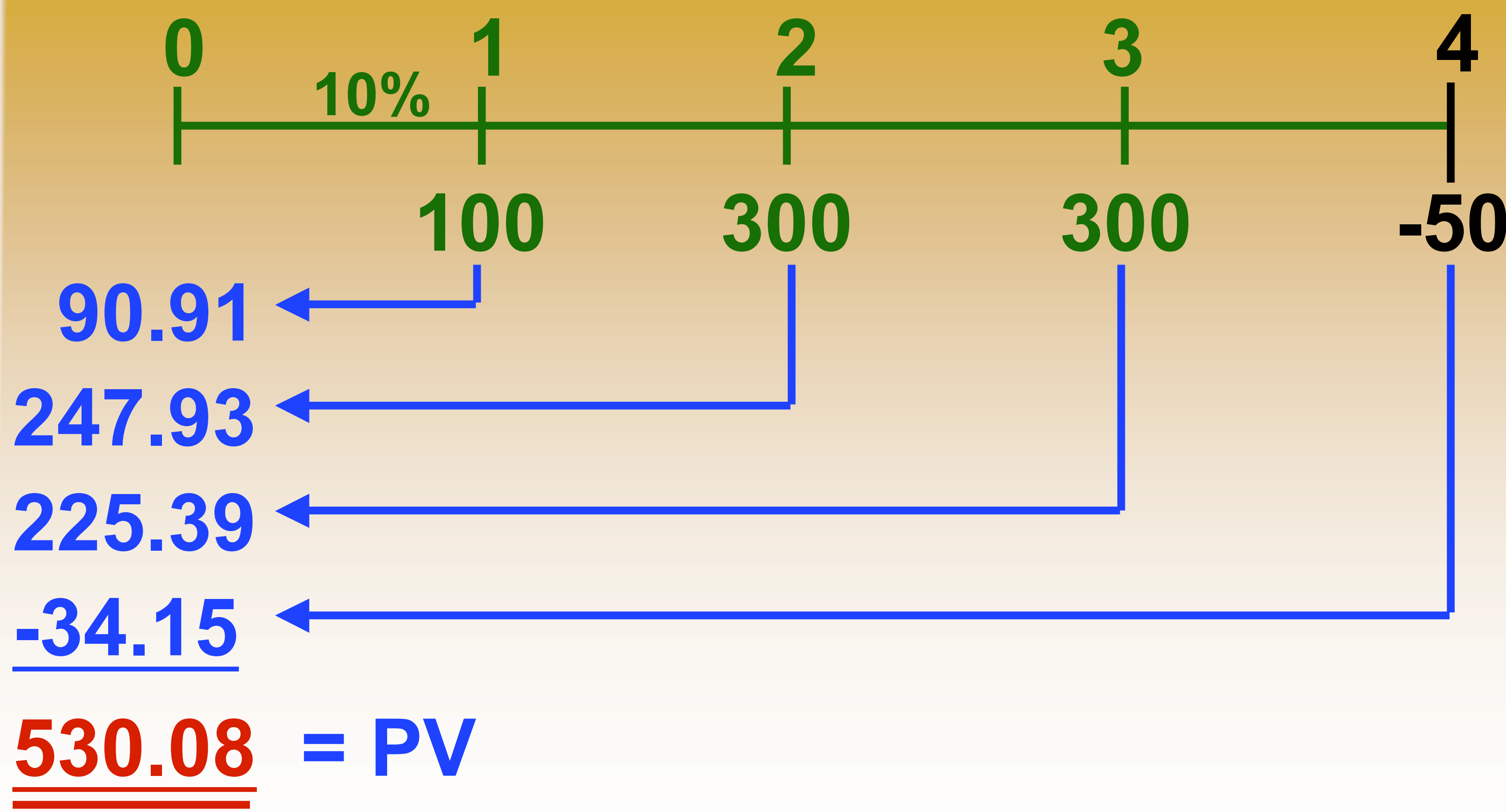
**Change the formula to:**

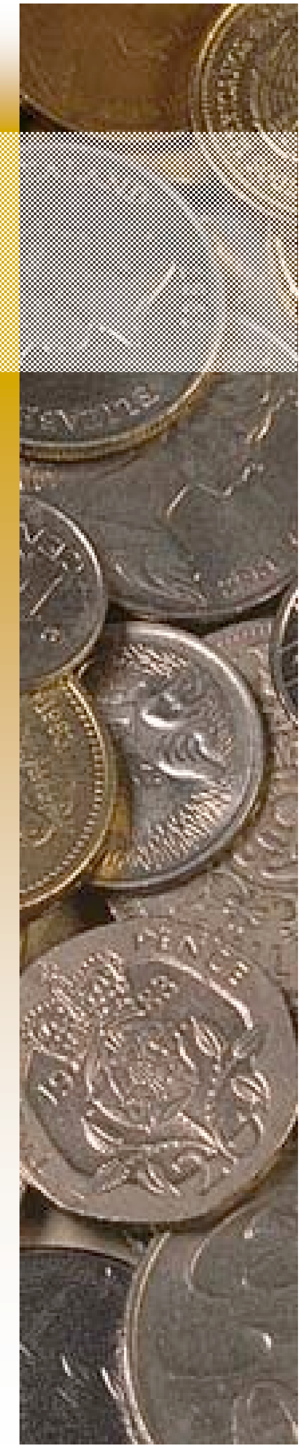
**=PV(10%,3,-100,0,1)**

**The fourth term, 0, tells the function there are no other cash flows. The fifth term tells the function that it is an annuity due. A similar function gives the future value of an annuity due:**

**=FV(10%,3,-100,0,1)**

# What is the PV of this uneven cashflow stream?



# Spreadsheet Solution

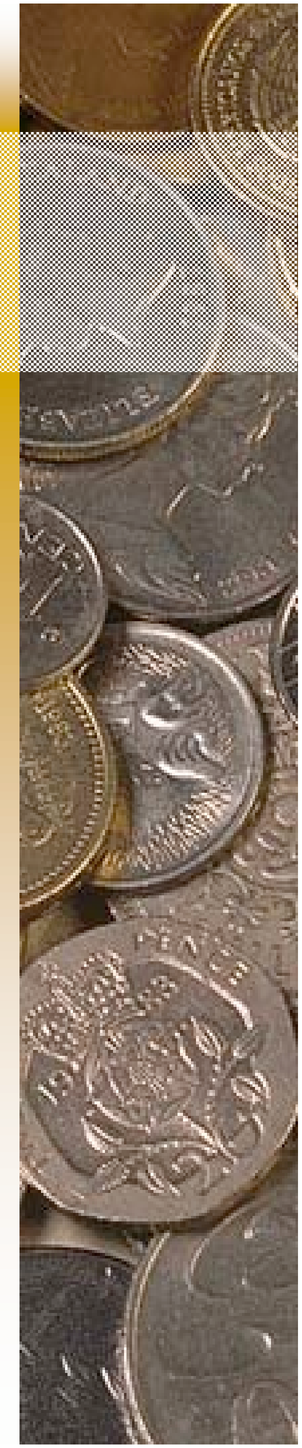
| | A | B | C | D | E |
|---|---|---|---|---|---|
| 1 | 0 | 1 | 2 | 3 | 4 |
| 2 | | 100 | 300 | 300 | -50 |
| 3 | 530.09 | | | | |

Excel Formula in cell A3:

=NPV(10%,B2:E2)

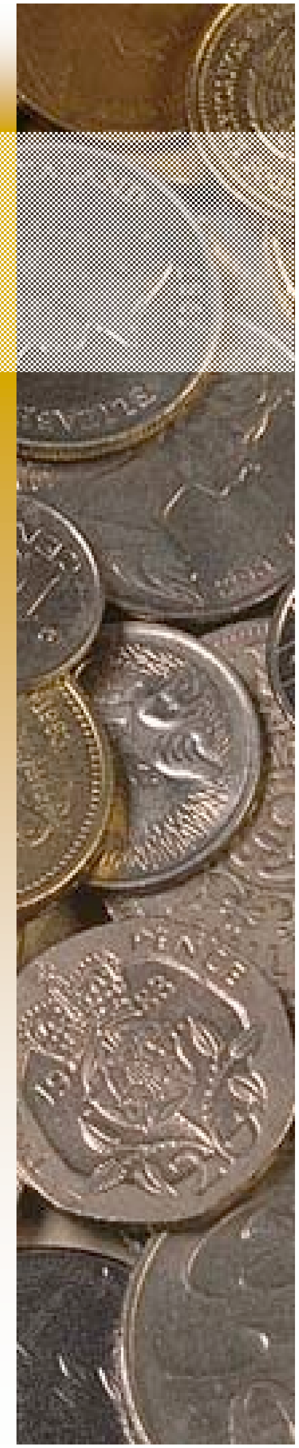
# Nominal rate ($i_{Nom}$)

- Stated in contracts, and quoted by banks and brokers.
- Not used in calculations or shown on time lines
- Periods per year (m) must be given.
- Examples:
  - 8%; Quarterly
  - 8%, Daily interest (365 days)

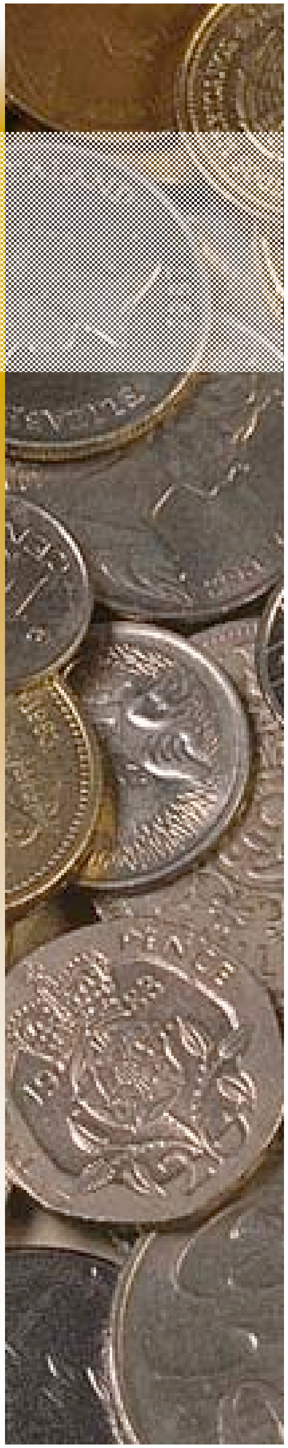


# Periodic rate ($i_{Per}$)

- $i_{Per} = i_{Nom}/m$, where m is number of compounding periods per year. m = 4 for quarterly, 12 for monthly, and 360 or 365 for daily compounding.
- Used in calculations, shown on time lines.
- Examples:
  - 8% quarterly: iPer = 8%/4 = 2%.
  - 8% daily (365): iPer = 8%/365 = 0.021918%.



## Will the FV of a lump sum be larger or smaller if we compound more often, holding the stated I% constant?  Why?



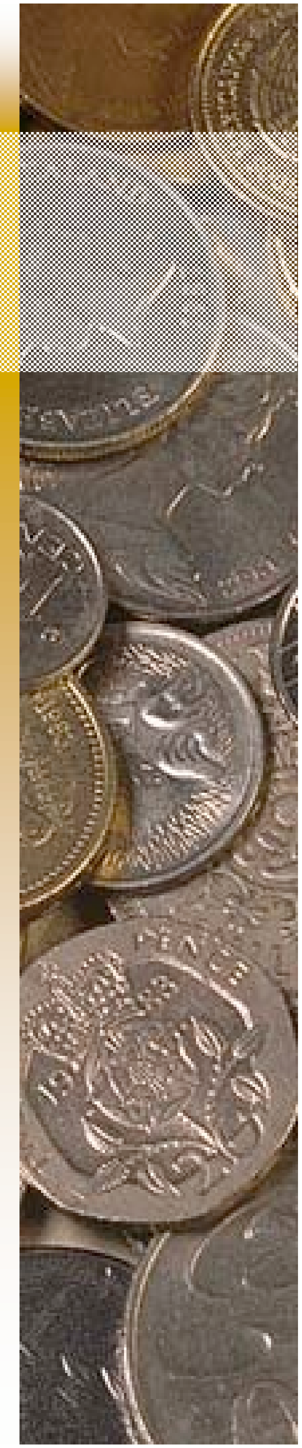
**LARGER!** If compounding is more frequent than once a year--for example, semiannually, quarterly, or daily--interest is earned on interest more often.

# FV Formula with Different Compounding Periods (e.g., $100 at a 12% nominal rate with semiannual compounding for 5 years)

$$FV_n = PV\left(1 + \frac{i_{Nom}}{m}\right)^{mn}.$$

$$FV_{5S} = \$100\left(1 + \frac{0.12}{2}\right)^{2x5}$$

$$= \$100(1.06)^{10} = \$179.08.$$



# FV of $100 at a 12% nominal rate for 5 years with different compounding

FV(Annual)= $\$100(1.12)^5 = \$176.23.$

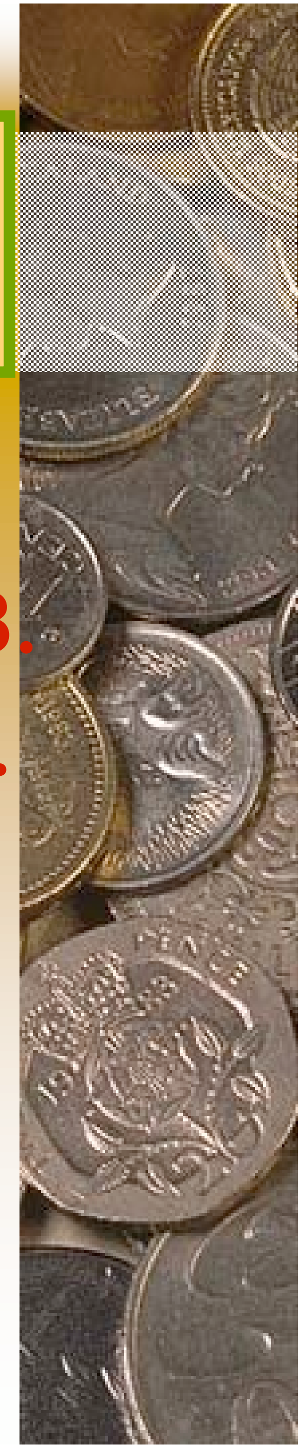
FV(Semiannual)= $\$100(1.06)^{10} = \$179.08.$

FV(Quarterly)= $\$100(1.03)^{20} = \$180.61.$

FV(Monthly)= $\$100(1.01)^{60} = \$181.67.$

FV(Daily)= $\$100(1+(0.12/365))^{(5 \times 365)}$

$= \$182.19.$



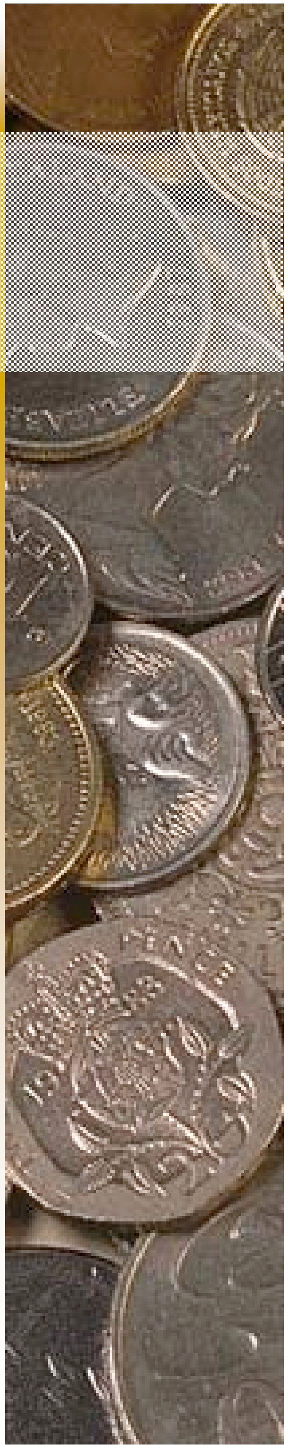
# Effective Annual Rate (EAR = EFF%)

- The EAR is the annual rate which causes PV to grow to the same FV as under multi-period compounding Example: Invest $1 for one year at 12%, semiannual:

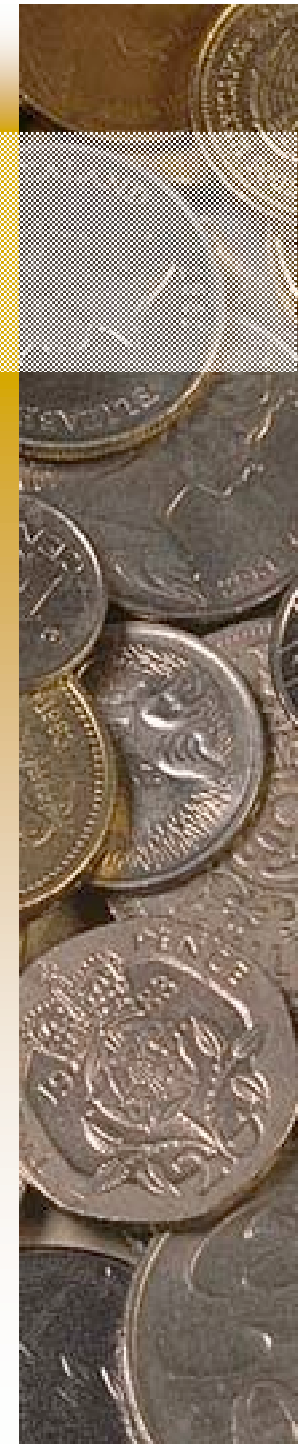
$$FV = PV(1 + i_{Nom}/m)^m$$

$$FV = \$1\ (1.06)^2 = 1.1236.$$

- EFF% = 12.36%, because $1 invested for one year at 12% semiannual compounding would grow to the same value as $1 invested for one year at 12.36% annual compounding.



- An investment with monthly payments is different from one with quarterly payments. Must put on EFF% basis to compare rates of return. Use EFF% only for comparisons.
- Banks say “interest paid daily.” Same as compounded daily.



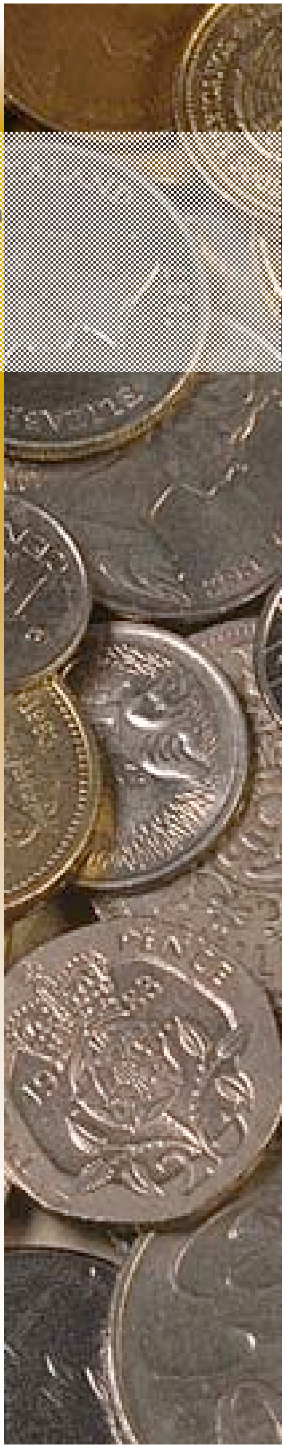
## How do we find EFF% for a nominal rate of 12%, compounded semiannually?

$$EFF\% = \left(1 + \frac{i_{Nom}}{m}\right)^m - 1$$

$$= \left(1 + \frac{0.12}{2}\right)^2 - 1.0$$

$$= (1.06)^2 - 1.0$$

$$= 0.1236 = 12.36\%.$$



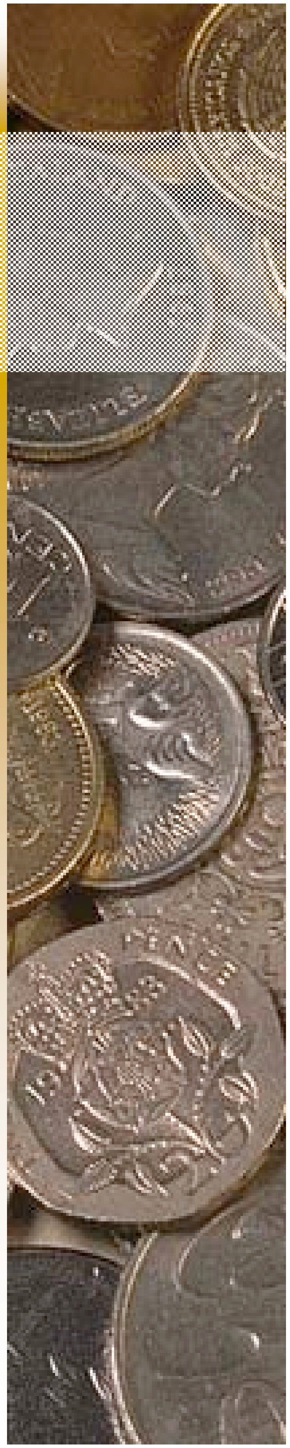
# EAR (or EFF%) for a Nominal Rate of of 12%

$EAR_{Annual} = 12\%.$

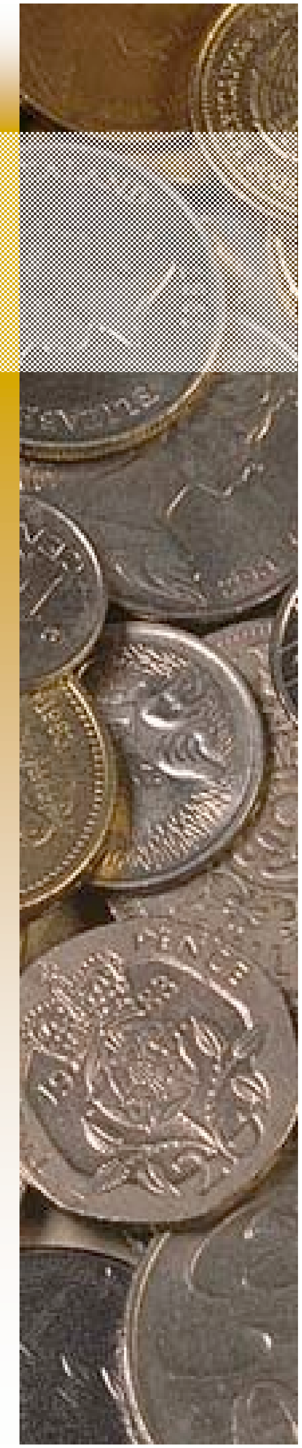
$EAR_{Q} = (1 + 0.12/4)^{4} - 1 = 12.55\%.$

$EAR_{M} = (1 + 0.12/12)^{12} - 1 = 12.68\%.$

$EAR_{D(365)} = (1 + 0.12/365)^{365} - 1 = 12.75\%.$



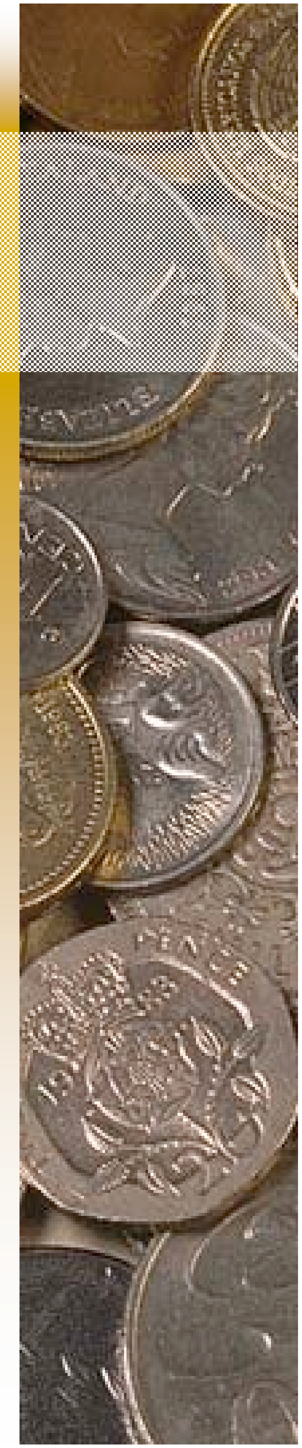
# Can the effective rate ever be equal to the nominal rate?



- Yes, but only if annual compounding is used, i.e., if m = 1.
- If m > 1, EFF% will always be greater than the nominal rate.

# When is each rate used?

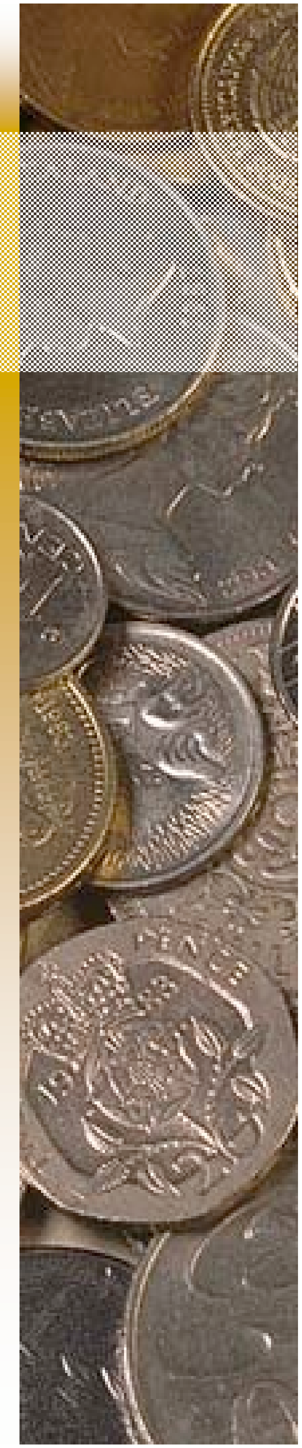
$i_{Nom}$: Written into contracts, quoted by banks and brokers. Not used in calculations or shown on time lines.



# When is each rate used?

$i_{Per}$: Used in calculations, shown on time lines.

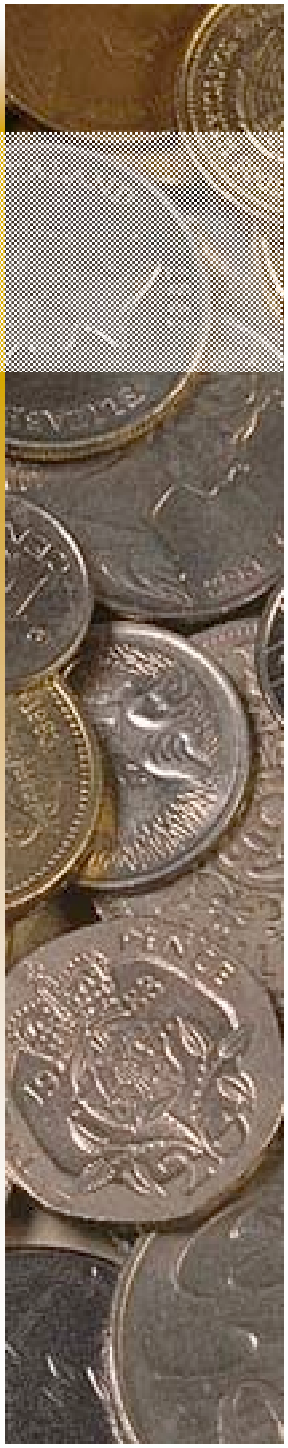
If $i_{Nom}$ has annual compounding, then $i_{Per} = i_{Nom}/1 = i_{Nom}$.



# EAR = EFF%:

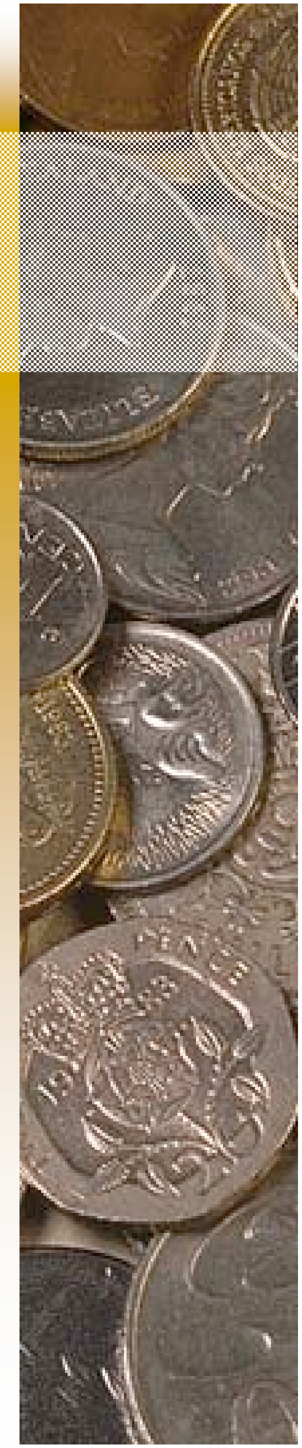
Used to compare returns on investments with different payments per year.

(Used for calculations if and only if dealing with annuities where payments don't match interest compounding periods.)



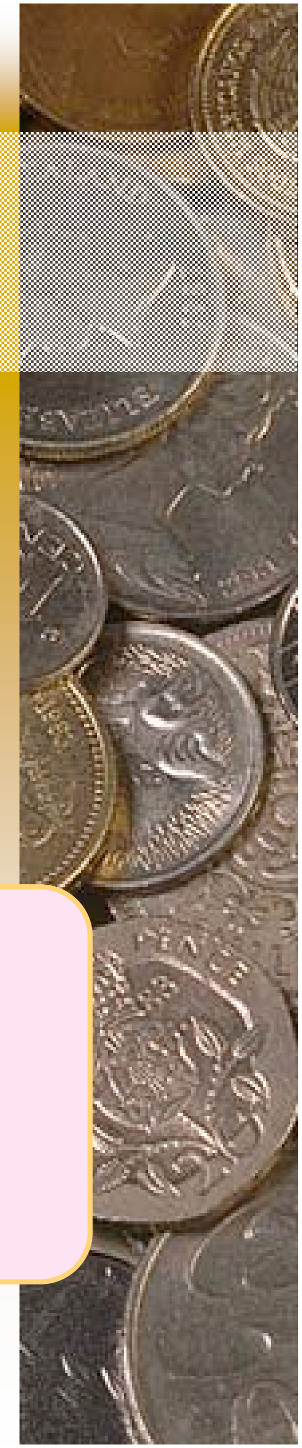
# Amortization

**Construct an amortization schedule for a $1,000, 10% annual rate loan with 3 equal payments.**



# Step 1: Find the required payments.


| | N | I/YR | PV | PMT | FV |
|---|---|---|---|---|---|
| INPUTS | 3 | 10 | -1000 | | 0 |
| OUTPUT | | | | 402.11 | |



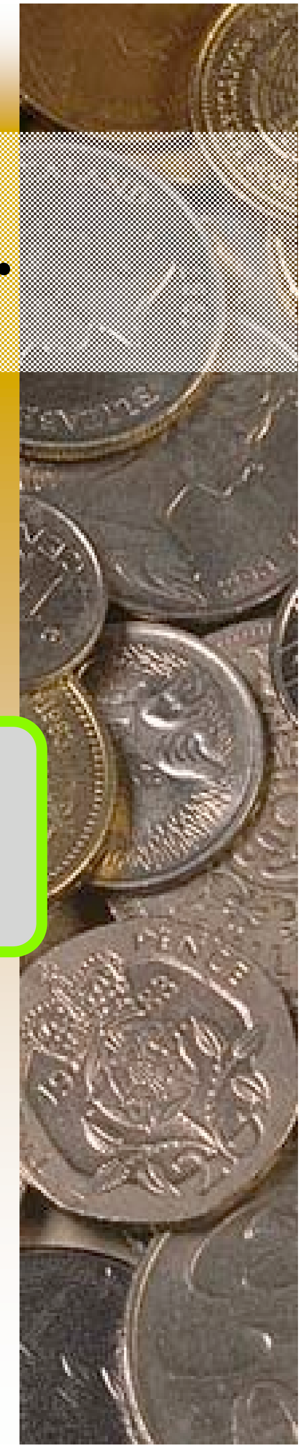
## Step 2: Find interest charge for Year 1.

$INT_t = \text{Beg bal}_t (i)$

$INT_1 = \$1{,}000(0.10) = \$100.$

## Step 3: Find repayment of principal in Year 1.

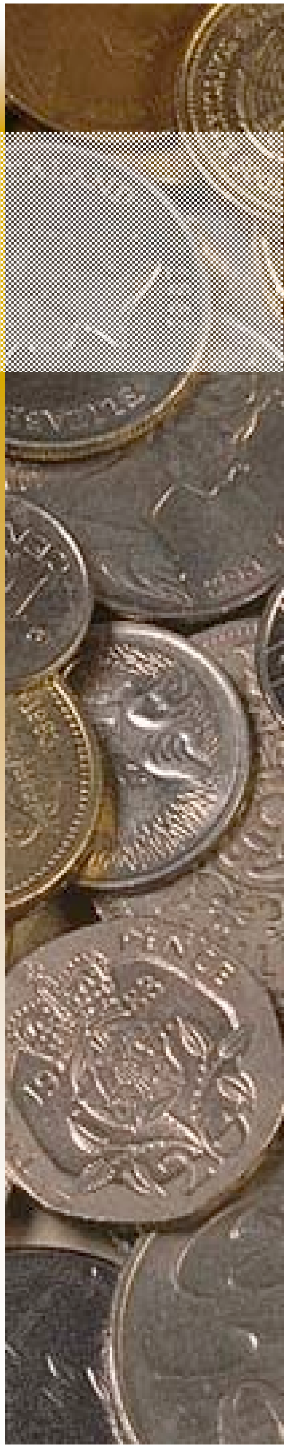
$$\begin{aligned}\text{Repmt} &= PMT - INT \\ &= \$402.11 - \$100 \\ &= \$302.11.\end{aligned}$$



## Step 4: Find ending balance after Year 1.

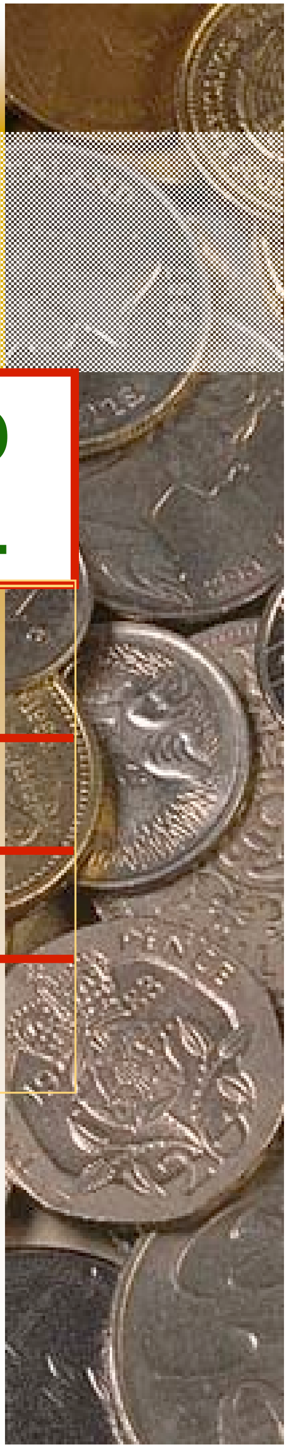
End bal = Beg bal - Repmt
= $1,000 - $302.11 = $697.89.

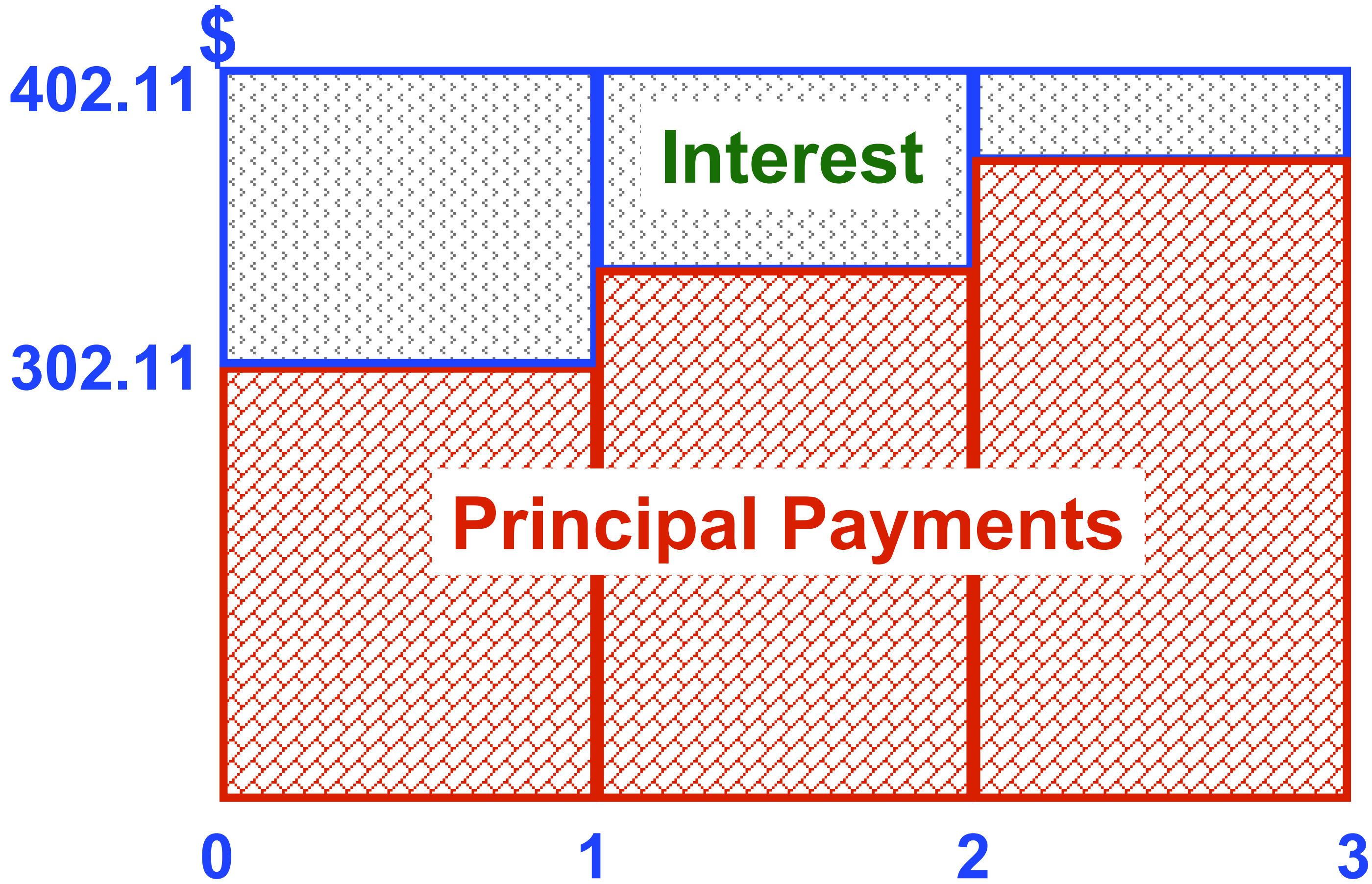
**Repeat these steps for Years 2 and 3 to complete the amortization table.**



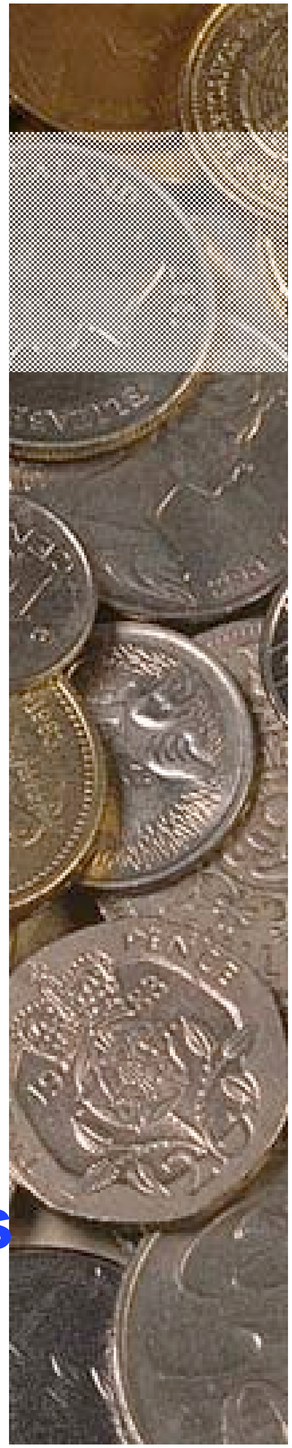
# Interest declines. Tax implications.

| YR | BEG BAL | PMT | INT | PRIN PMT | END BAL |
|---|---|---|---|---|---|
| 1 | $1,000 | $402 | $100 | $302 | $698 |
| 2 | 698 | 402 | 70 | 332 | 366 |
| 3 | 366 | 402 | 37 | 366 | 0 |
| TOT | | 1,206.34 | 206.34 | 1,000 | |

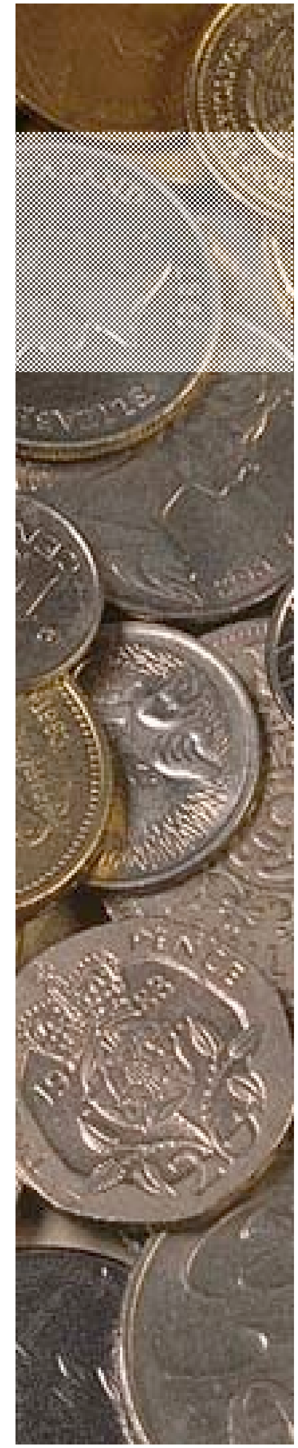





Level payments. Interest declines because outstanding balance declines. Lender earns 10% on loan outstanding, which is falling.

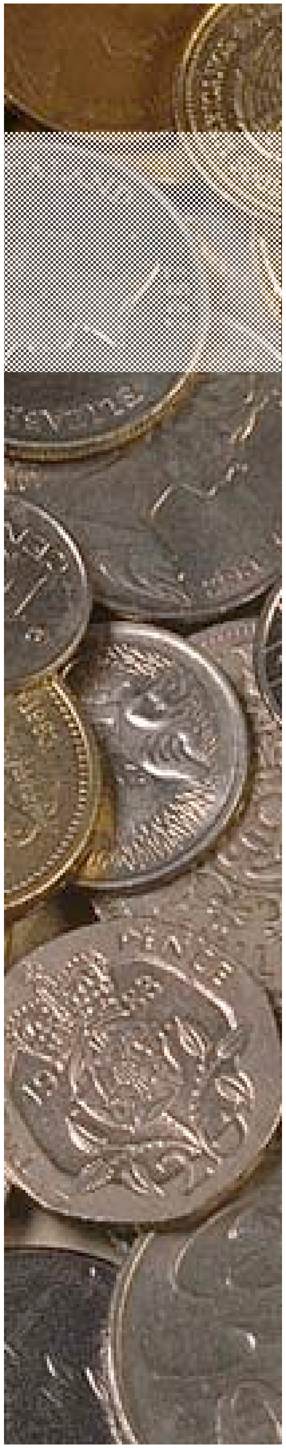


- Amortization tables are widely used--for home mortgages, auto loans, business loans, retirement plans, and so on.  They are very important!
- Spreadsheets are great for setting up amortization tables.



On January 1 you deposit $100 in an account that pays a nominal interest rate of 11.33463%, with daily compounding (365 days).

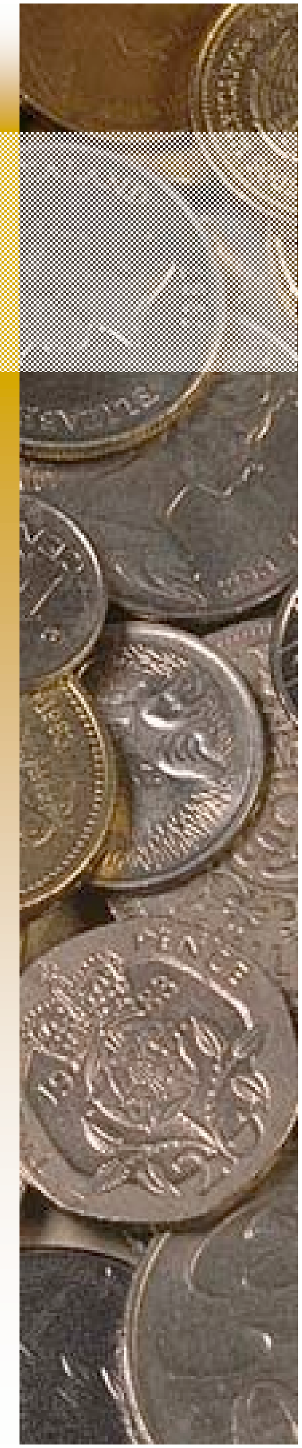
How much will you have on October 1, or after 9 months (273 days)? (Days given.)



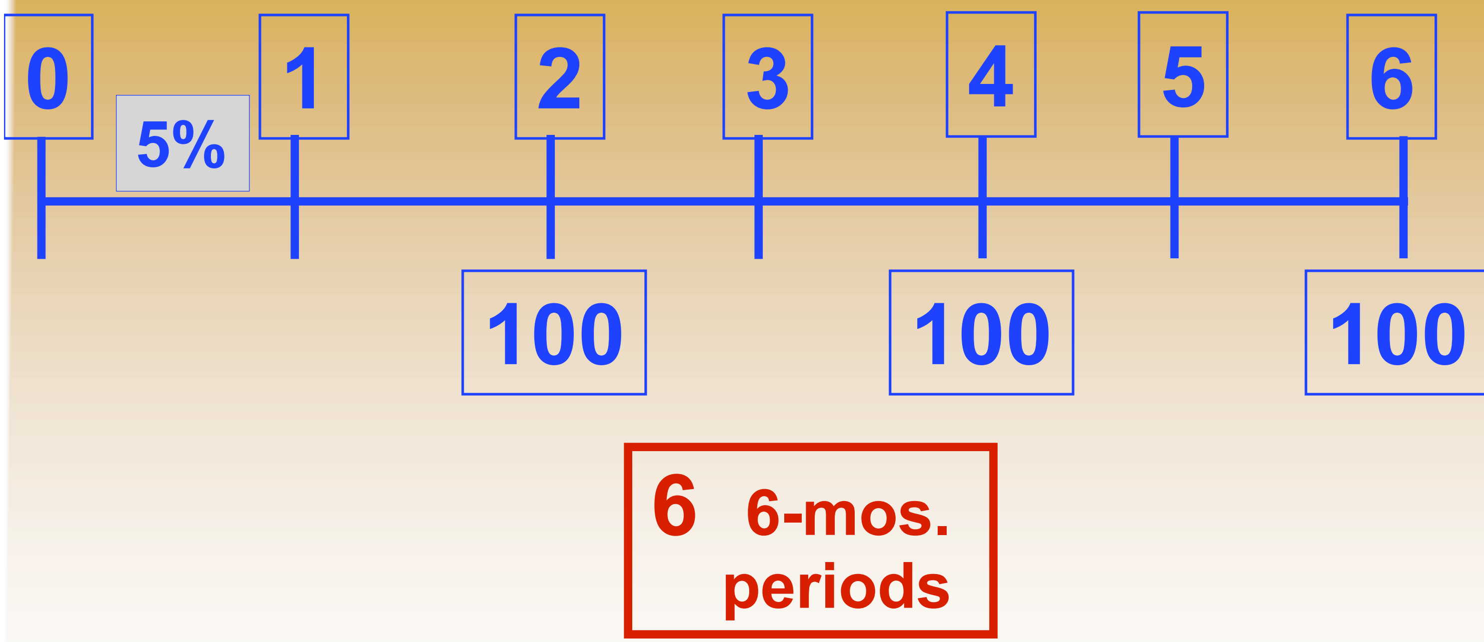
$$i_{Per} = 11.33463\%/365 = 0.031054\%\text{ per day.}$$

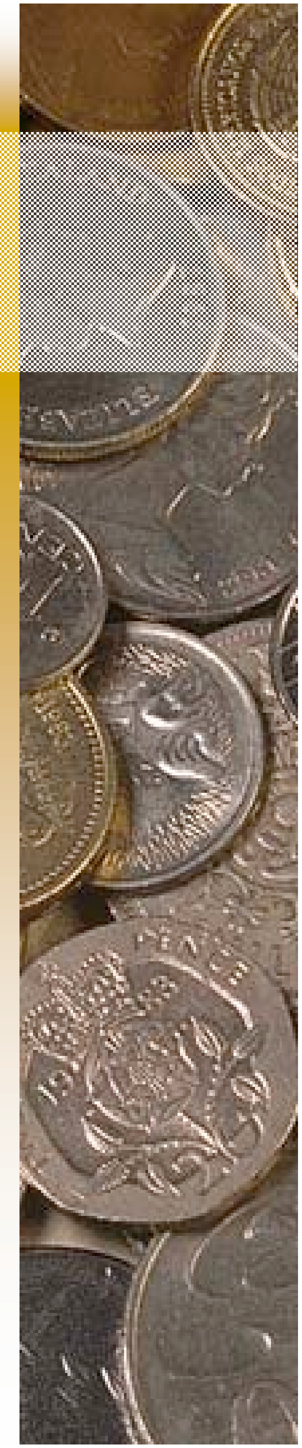

$$FV_{273} = \$100(1.00031054)^{273} = \$100(1.08846) = \$108.85.$$

Note: % in calculator, decimal in equation.

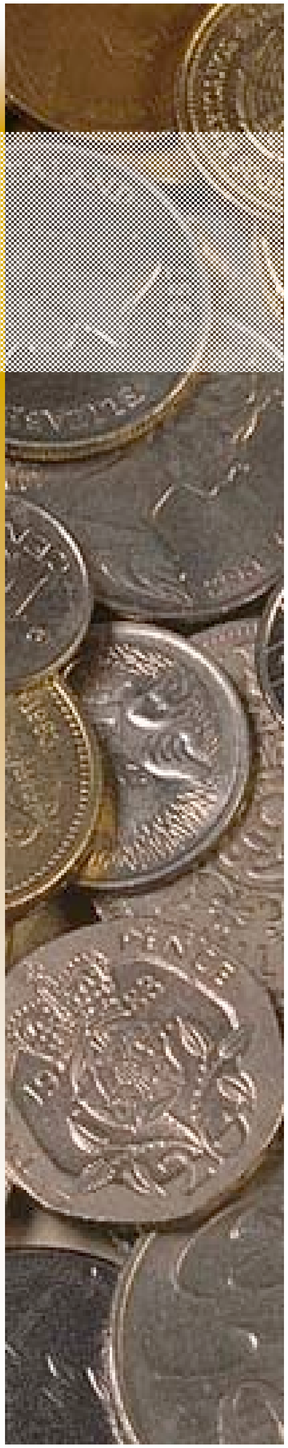


What's the value at the end of Year 3 of the following CF stream if the quoted interest rate is 10%, compounded semiannually?

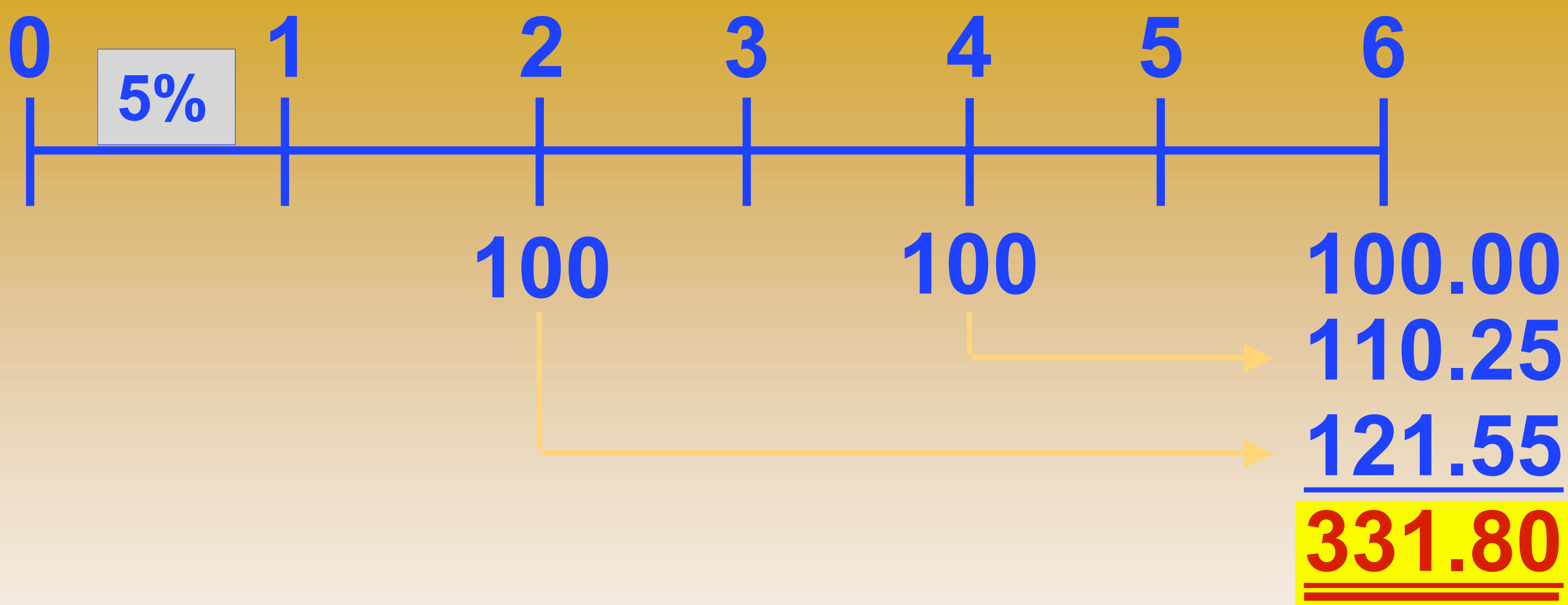


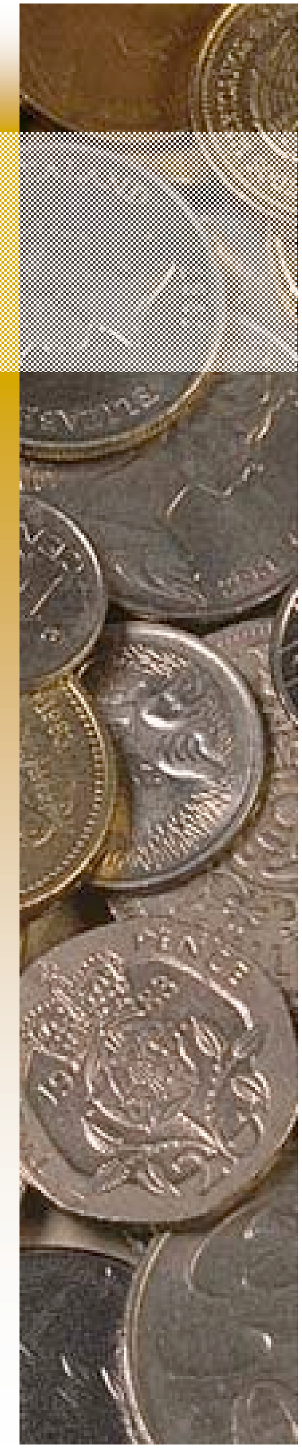
- Payments occur annually, but compounding occurs each 6 months.
- So we can’t use normal annuity valuation techniques.



# 1st Method: Compound Each CF




$$FVA_3 = \$100(1.05)^4 + \$100(1.05)^2 + \$100$$
$$= \$331.80.$$



# 2nd Method: Treat as an Annuity

$$\text{EAR} = \left(1 + \frac{0.10}{2}\right)^2 - 1 = 10.25\%.$$