Complexity Notation

*O notation:* We say that a function \( f(n) \) has a time complexity \( O(g(n)) \) if there exist constants \( c > 0 \) and \( n_0 \) such that
\[
0 \leq f(n) \leq cg(n) \text{ for } n \geq n_0
\]

*Ω notation:* We say that a function \( f(n) \) has a time complexity \( Ω(g(n)) \) if there exist positive constants \( c \) and \( n_0 \) such that
\[
0 \leq cg(n) \leq f(n) \text{ for } n \geq n_0
\]

*Θ notation:* We say that a function \( f(n) \) has a time complexity \( Θ(g(n)) \) if there exist positive constants \( c_1, c_2, \) and \( n_0 \) such that
\[
0 \leq c_1g(n) \leq f(n) \leq c_2g(n) \text{ for } n \geq n_0
\]

*o notation:* We say that a function \( f(n) \) has a time complexity \( o(g(n)) \) if for any positive constant \( c \), there exists an \( n_0 > 0 \) such that
\[
0 \leq f(n) < cg(n) \text{ for } n \geq n_0
\]

*ω notation:* We say that a function \( f(n) \) has a time complexity \( ω(g(n)) \) if for any positive constant \( c \), there exists an \( n_0 > 0 \) such that
\[
0 \leq cg(n) < f(n) \text{ for } n \geq n_0
\]

Limits

Assume that \( f(n) \) is asymptotically non-negative and \( g(n) \) is is asymptotically positive.

- \( \lim_{n \to \infty} f(n)/g(n) = d \) where \( 0 < d < \infty \) implies
  \[
  f(n) = O(g(n)), g(n) = O(f(n)), \text{ and } f = Θ(g)
  \]
- \( \lim_{n \to \infty} f(n)/g(n) = \infty \) implies
  \[
  g = O(f), f(n) \neq O(g(n)), f = Ω(g), \text{ and } f \neq Θ(g)
  \]
- \( \lim_{n \to \infty} f(n)/g(n) = 0 \) implies
  \[
  f(n) = O(g(n)), g(n) \neq O(f(n)), \text{ and } f \neq Θ(g)
  \]
- \( f = o(g(n)) \) if and only if \( \lim_{n \to \infty} f(n)/g(n) = 0 \)
- \( f = ω(g(n)) \) if and only if \( \lim_{n \to \infty} f(n)/g(n) = \infty \)
- Recall L’Hopital’s Rule: \( \lim_{n \to \infty} f(n)/g(n) = \lim_{n \to \infty} f'(n)/g'(n) \)
- Warning: Don’t always assume the converse. For example, \( f(n) = O(g(n)) \) does not necessarily imply that \( \lim_{n \to \infty} f(n)/g(n) = d \). Counterexample: let \( f(n) = n \) and \( g(n) = n(1 + \sin n) \). In this case, the limit doesn’t exist.