Physically Based Modeling

Objects obey physical laws, e.g. gravity, collisions, spring forces, etc.

Particle System

Each particle:

- position moves over time based on the forces acting on it, i.e. it obeys \( f = ma \).
- has 6 degrees of freedom: 3 position, 3 velocity

Equations of Motion:

Position: \( p = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \)

Velocity: \( v = \dot{p} = \frac{dp}{dt} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} \)

These 6 degrees of freedom are combined into a single vector \( u \) referred to as the phase space:

\[
    u = \begin{pmatrix} p \\ v \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \\ \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix}
\]

and where

\[
    \dot{u} = \begin{pmatrix} \dot{p} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} v \\ f/m \end{pmatrix}
\]

describes the path of the particle over time. Note,

\[
    \dot{v} = \text{acceleration} = a = \frac{f}{m}
\]

An equation of the form

\[
    \dot{u} = h(u, t)
\]

where \( h \) is some function, is referred to as a 1st order differential equation. If it can’t be solved exactly, then we solve it numerically.
Taylor’s Expansion

Taylor’s expansion says that

\[ u(t_0 + \Delta t) = u(t_0) + \Delta t \dot{u}(t_0) + \frac{(\Delta t)^2}{2!} \ddot{u}(t_0) + \ldots \]

It is exact but requires summing an infinite number of terms. For small \( \Delta t \) we can approximate using Euler’s Method

\[ u(t_0 + \Delta t) = u(t_0) + \Delta t \, h(u_0, t_0) + \Theta((\Delta t)^2) \]

Dropping the last term gives

\[ u(t_0 + \Delta t) = u(t_0) + \Delta t \, h(u_0, t_0) \]

This is iterated to obtain its value \( u \) at \( t_0, t_0 + \Delta t, t_0 + 2\Delta t, \ldots \) Often this is written as

\[ u_n = u_{n-1} + \Delta t \, h(u_{n-1}, t_{n-1}) \]

Euler’s method is the simplest method for approximating differential equations. A much better approximation is called the Midpoint or Runge Kutta Method given by

\[
\begin{align*}
    u_{n+1} & = u_n + k_2 + \Theta((\Delta t)^3) \\
    k_2 & = \Delta t \, h\left(u_n + \frac{1}{2}k_1, t_n + \frac{1}{2} \Delta t\right) \\
    k_1 & = \Delta t \, h(u_n, t_n)
\end{align*}
\]

Examples

1. Constant Motion

Assume that there are no forces, \( f = 0 \), so that \( a = 0 \) and \( v = v_c = \text{constant.} \) Then
\[ \dot{\mathbf{u}} = \begin{pmatrix} \dot{\mathbf{p}} \\ \dot{\mathbf{v}} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix} \]

Since \( \dot{\mathbf{p}} = \mathbf{v} \) is constant, we have that \( \ddot{\mathbf{p}} = \dddot{\mathbf{p}} = \ldots = \mathbf{0} \). Putting this into Taylor’s Expansion gives

\[ u(t_0 + \Delta t) = u(t_0) + \Delta t \ h(u_0, t_0) = u(t_0) + \Delta t \begin{pmatrix} \mathbf{v} \\ \mathbf{0} \end{pmatrix} \]

or

\[ p(t_0 + \Delta t) = p(t_0) + \Delta t \ \mathbf{v_c} \]

which is what one expects for constant velocity. This is exact - no approximation is required.

Iterating, we have

\[ u_n = \begin{pmatrix} p_n \\ v_n \end{pmatrix} = \begin{pmatrix} p_{n-1} \\ v_c \end{pmatrix} + \Delta t \begin{pmatrix} \mathbf{v_{c, x}} \\ \mathbf{v_{c, y}} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} p_0 \\ v_c \end{pmatrix} + n \ \Delta t \begin{pmatrix} \mathbf{v_{c, x}} \\ \mathbf{v_{c, y}} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix} \]

Although iterating is hardly necessary since equation (1) is so simple.

2. Gravity

Assume the force is a constant in negative \( y \)-direction

\[ a = \dot{\mathbf{v}} = \frac{\mathbf{f}}{m} = \begin{pmatrix} \mathbf{0} \\ -g \\ \mathbf{0} \end{pmatrix} \]

so that

\[ \dot{\mathbf{u}} = \begin{pmatrix} \mathbf{v} \\ \frac{\mathbf{f}}{m} \end{pmatrix} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ 0 \\ -g \\ 0 \end{pmatrix} \]

Applying Euler’s Method, gives

\[ u_n = u_{n-1} + \Delta t \ h(u_{n-1}, t_{n-1}) \]
\[ u_n = \left( \begin{array}{c} p_n \\ v_n \end{array} \right) = \left( \begin{array}{c} x_{n-1} \\ y_{n-1} \\ z_{n-1} \\ v_{x,n-1} \\ v_{y,n-1} \\ v_{z,n-1} \end{array} \right) + \Delta t \left( \begin{array}{c} v_{x,n-1} \\ v_{y,n-1} \\ v_{z,n-1} \\ 0 \\ -g \\ 0 \end{array} \right) \]

3. Springs

[Diagram of springs with forces p and q]

Hooke’s Law gives
\[ f = -k_s |d| s \frac{d}{|d|} \]
where
\[ k_s = \text{spring constant} \]
\[ d = p - q = \text{direction of force} \]
\[ s = \text{spring resting length} \]

So we have
\[ a = \dot{v} = \frac{f}{m} = -k_s |d| s \frac{d}{|d|} \]
and
\[ \dot{u} = \left( \begin{array}{c} v \\ \frac{f}{m} \end{array} \right) = \left( \begin{array}{c} \dot{x} \\ \dot{y} \\ \dot{z} \\ \frac{f_x}{m} \\ \frac{f_y}{m} \\ \frac{f_z}{m} \end{array} \right) \]

Applying Euler’s Method, gives
\[ u_n = u_{n-1} + \Delta t \ h(u_{n-1}, t_{n-1}) \]
\[ u_n = \left( \begin{array}{c} p_n \\ v_n \end{array} \right) = \left( \begin{array}{c} x_{n-1} \\ y_{n-1} \\ z_{n-1} \\ v_{x,n-1} \\ v_{y,n-1} \\ v_{z,n-1} \end{array} \right) + \Delta t \left( \begin{array}{c} v_{x,n-1} \\ v_{y,n-1} \\ v_{z,n-1} \\ \frac{f_x}{m} \\ \frac{f_y}{m} \\ \frac{f_z}{m} \end{array} \right) \]