MATH 251

Midterm Exam 3

April 21, 2003

NAME (please print legibly): ____
Your University ID Number: ____

- **No books, notes, note cards, or calculators are allowed on this exam.**
- **Please write in complete sentences. You may use blackboard shorthand, i.e. $\exists$ and $\forall$ to save time. You may use back pages if necessary. You may not receive full credit for a correct answer if there is no work shown.**
- **Note: Some of these problems are similar to homework problems. You will receive no points for stating "This is true by a homework problem."**

QUESTION	VALUE	SCORE
1	25	
2	25	
3	25	
4	25	
5	25	
TOTAL	125	

List of Theorems

You may use (without proving) the following theorems in your proofs on this exam. Make sure to cite the theorem when you use it, e.g. (by Theorem ** we have that ...).

Theorem 1 Let A and B be non-empty sets, and let $f : A \to B$ be a function.

(i) The function f has a right inverse iff f is surjective.

(ii) The function f has a left inverse iff f is injective.

(iii) The function f has an inverse iff f is bijective.

Theorem 2 Let A be a non-empty set. The following are equivalent.

(i) The set A is countable.

(ii) The set A is a subset of a countable set.

(iii) There is an injective map $f : A \to \mathbb{N}$.

(iv) There is a surjective map $f : \mathbb{N} \to A$.

Theorem 3 (The Division Algorithm)

Let $a, b \in \mathbb{Z}$ and suppose $b \neq 0$. Then there are unique $q, r \in \mathbb{Z}$ such that $a = b \cdot q + r$ where $0 \leq r < |b|$.

Theorem 4 Let A, B be finite sets.

(i) If $X \subseteq A$, then X is finite.

(ii) If $X \subseteq A$, then $|A| = |X| + |(A - X)|$.

(iii) If $X \subsetneqq A$, $|X| < |A|$.

(iv) If $X \subsetneqq A$, $X \sim A$.

Theorem 5 (Schroeder-Bernstein Theorem)

Let A and B be sets. Suppose that $A \preceq B$ and $B \preceq A$. Then $A \sim B$.

Instructions. Do any **four** of the following five problems. DO NOT DO ALL FIVE. If you do all five only the first four will be graded.

1. (25 pts) Let A, B and C be sets. Prove or give a counterexample to each the following statements.

(i) Suppose $A \cup C \sim B \cup C$, then $A \sim B$.

(ii) Suppose $A \sim B$ and that $A \cap C = \emptyset = B \cap C$. Prove that $A \cup C \sim B \cup C$.

2. (25 pts) Let A be a set. Consider the function $\Phi : \mathcal{P}(A) \to \mathcal{P}(A)$ defined by

$\Phi(X) = A - X$ for all $X \subseteq A$. Prove that Φ is bijective.

3. (25 pts) Let $A = \{0, 1, 2\} \cup [3, 7)$ and let $B = [-2, 5] \cup \{6, 7\}$. Prove $A \sim B$.

4. (25 pts) Let R be a symmetric and transitive relation on A. Define the relation R' on A by $R' = A \times A - R$.

(i) Prove or give a counterexample to the statement; R' is symmetric.

(ii) Prove or give a counterexample to the statement; R' is transitive.

5. (25 pts) Prove or give a counterexample to each of the proposed cancelation laws.

Let $a, b, c \in \mathbb{Z}$ and let $n \in \mathbb{N} \setminus \{0\}$.

(i) $a + c \equiv b + c \pmod{n}$ implies $a \equiv b \pmod{n}$.

(ii) $ac \equiv bc \pmod{n}$ implies $a \equiv b \pmod{n}$.