Math 253

Assignment 13, VERY LAST ASSIGNMENT!!

Please do the following problems:

1. Determine which of the following are linear transformations. If T is a linear transformations give a quick proof. If T isn't linear show why via an explicit calculation.

   (a) $T : \mathbb{R}^4 \to \mathbb{R}^2$, given by $T(a, b, c, d) = (ab + d, c)$.

   (b) $T : \mathbb{R}^4 \to \mathbb{R}^2$, given by $T(a, b, c, d) = (a - 2b + d, c)$.

   (c) $T : M_{2\times 2}(R) \to M_{2\times 2}(R)$, given by $T(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = \begin{bmatrix} a+d & b-c \\ c & 0 \end{bmatrix}$

2. For each of the linear transformations T above in problem 1. Find the matrix representation of the linear transformation.

   Note: The standard basis for $M_{2\times 2}(R)$ is

$$\epsilon = \{\vec{e_1}, \vec{e_2}, \vec{e_3}, \vec{e_4}\} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

3. Consider the following basis for $\mathbb{R}^2$.

$$\beta = \{\vec{v_1}, \vec{v_2}\} \qquad \text{where } \vec{v_1} = (-1, 2) \text{ and } \vec{v_2} = (1, 3)$$

   (a) Calculate $4\vec{v_1} - 1\vec{v_2}$.

   (b) Recall, if $\vec{w} = a\vec{v_1} + b\vec{v_2}$, then ${}_\beta[\vec{w}] = (a, b)$. Suppose $\vec{w} = (-5, 3)$ find ${}_\beta[\vec{w}]$. Hint: use part (a).

   (c) If $\vec{u} = (-3, 6)$, find ${}_\beta[\vec{u}]$.

   (d) If $\vec{q} = (-3, -9)$, find ${}_\beta[\vec{q}]$.

   (e) Find the matrix, ${}_\epsilon[I]_\beta$, which changes coordinates from β coordinates to standard ϵ coordinates. Note, here $\epsilon = \{(1, 0), (0, 1)\} = \{\vec{e_1}, \vec{e_2}\}$.

   (f) Find the matrix, ${}_\beta[I]_\epsilon$, which changes coordinates from ϵ coordinates to β coordinates.

   (g) Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation such that $T(\vec{v_1}) = 2\vec{v_2}$ and $T(v_2) = -1\vec{v_1}$. Find the matrix representation for this linear transformation T as follows; first find ${}_\beta[T]_\beta$, then find ${}_\epsilon[T]_\epsilon = {}_\epsilon[I]_\beta {}_\beta[T]_\beta {}_\beta[I]_\epsilon$

   (h) Find $T(7, 3)$. Find a formula for $T(a, b)$.

4. **Optimization of Quadratic Forms** (a) Find the change of coordinates matrix Q such that $\vec{x} = Q\vec{y}$ that transforms the quadratic form $\vec{x}^T A\vec{x}$ into $\vec{y}^T \Lambda\vec{y}$.

$$3x_1^2 + 2x_2^2 + 2x_3^2 + 2x_1x_2 + 2x_1x_3 + 4x_2x_3 = 5y_1^2 + 2y_2^2 + 0y_3^2$$

   (b) Let $F(\vec{x}) = 7x_1^2 + x_2^2 + 7x_3^2 - 8x_1x_2 - 4x_1x_3 - 8x_2x_3$.

   Find the Maximum, M, and minimum, m, values of the quadratic form $F(\vec{x})$ subject to the constraint $\vec{x}^T\vec{x} = 1$.

   Find unit vectors where the max and min values are attained.

5. Find a singular value decomposition for the matrix $A = \begin{bmatrix} 7 & 1 \\ 0 & 0 \\ 5 & 5 \end{bmatrix}$. Check your answer by multiplying the matrices $U\Sigma V^T$.

6. If A has singular value decomposition $U\Sigma V^T$, describe a singular value decomposition for A^T. If A is invertible what does this say about $U\Sigma V^T$? What would a singular value decomposition for A^{-1} look like?