An Introduction to Graph Automorphism Groups

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Definitions

Definition
An **isomorphism** between two simple graphs $G$ and $G'$ is a bijection, $\phi : V(G) \to V(G')$ such that for all $u, v \in V(G)$, $uv \in E(G)$ if and only if $\phi(u)\phi(v) \in E(G')$. 
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The primary difference is that in an isomorphism, $G'$ can be a different drawing of $G$, but in an automorphism, $G$ must be drawn the same way every time.
Example of an Automorphism
Example of an Automorphism

Consider $\phi$

\[
\begin{array}{c|ccccc}
v & a & b & c & d & e \\
\phi(v) & c & d & e & a & b \\
\end{array}
\]
Example of an Automorphism

Consider \( \phi \)

\[
\begin{array}{ccccc}
\phi(v) & a & b & c & d & e \\
\hline
v & c & d & e & a & b \\
\end{array}
\]

Edges Preserved

\[
\begin{array}{c}
\overline{ab} & \overline{bc} & \overline{cd} & \overline{de} & \overline{ea} \\
\hline
\overline{cd} & \overline{de} & \overline{ea} & \overline{ab} & \overline{bc} \\
\end{array}
\]
Example of an Automorphism

Consider \( \phi \)

\[
\begin{array}{cccccc}
\mathcal{V} & a & b & c & d & e \\
\phi(\mathcal{V}) & c & d & e & a & b \\
\end{array}
\]

Edges Preserved

\[
\begin{array}{cccccc}
\overline{ab} & \overline{bc} & \overline{cd} & \overline{de} & \overline{ea} \\
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\]
Example of an Automorphism

Question
How many distinct automorphisms exist on $C_5$?
Example of an Automorphism

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How many distinct automorphisms exist on $C_5$?

Answer
10
Example of an Automorphism

Notation:
We define $r$ as a rotation. This maps $a$ to $b$, and $b$ to $c$, etc. Thus, $\phi$ in this example was $r^2$. 
Notation:
We define $f$ as a flip, or reflection over the vertical axis. This keeps vertex $a$ where it is, switches vertex $b$ with vertex $d$ etc.
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Automorphisms of $C_5$

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Properties of Automorphisms

If $\phi$ maps every vertex to itself, $\phi$ is an automorphism. We call $\phi$ the *identity*. 
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maps to

maps to
Properties of Automorphisms

If $\phi$ is an automorphism, and $\psi$ is the map that reverses $\phi$, i.e. for all $a \in V(G)$, $\psi(\phi(a)) = a$, then $\psi$ is an automorphism.
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- Note that because $\phi$ is a bijection, it has an inverse, and $\psi$ is that inverse.
Properties of Automorphisms

If $\phi$ is an automorphism, and $\psi$ is the map that reverses $\phi$, i.e. for all $a \in V(G)$, $\psi(\phi(a)) = a$, then $\psi$ is an automorphism.

▶ Note that because $\phi$ is a bijection, it has an inverse, and $\psi$ is that inverse.

▶ The important thing about $\psi$ or $\phi^{-1}$ is that it maintains the automorphism property.
Properties of Automorphisms

\( \phi^{-1} \) is an automorphism.

\( \phi \) maps to
Properties of Automorphisms

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\( \phi^{-1} \) maps to
Properties of Automorphisms

$\phi^{-1}$ is an automorphism.
If $\psi$ and $\phi$ are both automorphisms, the *composition* $\psi \circ \phi$ is also an automorphism.
Properties of Automorphisms

If \( \psi \) and \( \phi \) are both automorphisms, the *composition* \( \psi \circ \phi \) is also an automorphism.

\[
\begin{array}{c}
\text{Maps to}
\end{array}
\]
Properties of Automorphisms

If \( \psi \) and \( \phi \) are both automorphisms, the \textit{composition} \( \psi \circ \phi \) is also an automorphism.

\[ f \text{ maps to} \]

\[
\begin{array}{c}
\text{Diagram 1} \\
\text{Diagram 2}
\end{array}
\]
Properties of Automorphisms

If \( \psi \) and \( \phi \) are both automorphisms, the composition \( \psi \circ \phi \) is also an automorphism.

\( rf \) maps to
Properties of Automorphisms

- The *identity* is an automorphism.
- The *inverse* $\phi^{-1}$ of an automorphism $\phi$ is an automorphism.
- The *composition* of two automorphisms is an automorphism.
Groups!

Definition

A *group* is a set $S$ and binary operation $\ast$, with these properties:

1. The operation $\ast$ is associative, i.e. for all $a, b, c \in S$,
   $$(a \ast b) \ast c = a \ast (b \ast c).$$
2. $S$ is closed under $\ast$, or in other words for all $a, b \in S$,
   $$a \ast b \in S.$$
3. There is an identity element $e \in S$ such that for all $a \in S$,
   $$a \ast e = e \ast a = a.$$
4. $S$ is closed under $\ast$. That is, for every $a \in S$ there exists an element
   $a^{-1}$ such that
   $$a \ast a^{-1} = e = a^{-1} \ast a.$$
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- $S$ is closed under $\star$. That is, for every $a \in S$ there exists an element $a^{-1}$ such that $a \star a^{-1} = e = a^{-1} \star a$.

We say that $S$ is a group under $\star$.
Example of a Group: $\mathbb{Z}_4$

The set $S$ is the set of equivalence classes modulo 4, with operation addition.

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**Theorem**
For every graph, the set of automorphisms forms a group under composition.
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Proof
We define our set $S$ to be the set of automorphisms for a graph $G$, and the operation, $\star$ is composition of automorphisms, which is the same as function composition, so we will denote it as $\circ$. By definition, function composition is associative. Because we have shown that if $\phi$ and $\psi$ are automorphisms, then $\phi \circ \psi$ is also an automorphism, it is clear that $S$ is closed. In addition, from the previous slide, $S$ clearly includes an identity. Finally, we also demonstrated that the inverse of every element is in $S$, so it is clear that $S$ is a group by definition.
### Automorphism Group for $C_5$

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Bowtie Automorphism Group

THE BOWTIE!
### Bowtie Automorphism Group

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- **$e$**: Do nothing
- **$L$**: Switch two vertices on left side
- **$T$**: Switch top vertices
- **$D$**: Switch top left vertex with bottom right
- **$R$**: Switch two vertices on right side
Ubiquitous Group Automorphism Theorem

Theorem
Every group is the automorphism group of a simple undirected graph
Definition
Given a group $S$, the generating set $H$ a set of elements, such that every element in the group is a product of elements in that set. For example, in $D_5$, which is the automorphism group for $C_5$, the generating set $H = \{r, f\}$. 
Definition
Given a group $S$, the generating set $H$ a set of elements, such that every element in the group is a product of elements in that set. For example, in $D_5$, which is the automorphism group for $C_5$, the generating set $H = \{ r, f \}$.

Definition
Given a group $G$ the **Cayley Graph** for $G$ is a directed graph with a vertex for each element of the group constructed with this rule:
For each ordered pair of vertices $a, b$ construct directed edge $ab$ if there exists an element $x \in S$ such that $xa = b$, where $S$ is a generating set of $G$. Label this edge with the corresponding generating set element $x$. 
The Group $D_3$

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Note that the group $D_n$ is defined to be the automorphism group for the graph $C_n$. However, the Cayley Graph is an algorithm for any group, so is a little bit more complex.
Definition
Given a Cayley Graph, the Frucht Graph is the graph obtained by replacing each edge with a gadget, with distinct gadgets only for distinctly labeled edges.
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Given a Cayley Graph, the **Frucht Graph** is the graph obtained by replacing each edge with a **gadget**, with distinct gadgets only for distinctly labeled edges.

Definition
A **gadget** is a simple graph that:

1. Has no automorphisms beyond the identity
2. Is not isomorphic to the original graph.
3. Encodes a direction, that is, it has no ‘flip’ automorphism.
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Example
Theorem
Every group is the automorphism group of a simple undirected graph
Ubiquitous Group automorphism Theorem

Theorem
Every group is the automorphism group of a simple undirected graph

Proof:
Let $G$ be a group, and let $S$ be a generating set for $G$. Construct the Cayley Graph of $G$ from $S$. It is clear that the Cayley graph has an automorphism group of precisely $G$, because of the limits set upon the edge construction. Then, construct the Frucht graph from the Cayley graph. Because the gadgets contain no internal automorphisms, this does not add any automorphisms to the existing automorphism group of our graph. Thus, there is now a simple undirected graph for which $G$ is the automorphism group.
Thank You for Listening
Sources

