The Game of Cops and Robbers on Graphs

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The Game

- Cops and Robbers is a game played on a reflexive graph (the vertices each have at least one loop). There are two players: a set of cops C, and a single robber R.
- The cops and robber occupy vertices.
- The cops go first in round 0.
- When a player moves in a round, they must move to a neighboring vertex. The loops allow a player to stay on the same vertex.
- Any subset of C may move in a given round.
Winning the Game

- The cops win if after some finite number of rounds, one of them can occupy the same vertex as the robber. This is called a capture.
- The robber wins if he can evade capture indefinitely.
- The minimum number of cops required to win in a graph $G$ is called the cop number of the graph $G$, denoted $c(G)$. If $c(G) = k$, we say $G$ is $k$-cop-win. When $k = 1$, we say $G$ is cop-win.
History

- The game was first considered in 1978 by Quilliot, a doctoral student at the University of Paris.
- In 1983, Nowakowski and Winkler wrote a paper independent from Quilliot's research.
- In 1984, Aigner and Fromme were the first to consider the game with multiple cops, introducing the idea of the cop number.
- Since then, several variations of the game have been introduced.
Lower bounds

1. For $n > 0$, $c(P_n) = c(W_n) = c(K_n) = 1$.
2. For $n \geq 4$, $c(C_n) = 2$.
3. $c(T) = 1$ if $T$ is a tree.

Proof: Place the cop on an arbitrary vertex of $T$. On each subsequent round move the cop directly toward the robber along the unique path between them. Eventually the robber will occupy a leaf since $T$ is a finite tree. He will be captured soon after.
Theorem: $c(G) \leq \gamma(G)$, where $\gamma$ is the domination number of $G$.

Proof: Let $G$ be a graph with domination number $\gamma$, and find a minimum dominating set $X$ of $G$.

In round 0, place a cop on each vertex of $X$. Since $X$ is dominating, then every vertex of $G$ is in $X$ or adjacent to some vertex of $X$. So the cops win in round 0 or 1.
Corollary: $c(\text{Pete}) \leq \gamma(\text{Pete}) = 3$
Theorem (Aigner and Fromme, 1984)

If $G$ has girth at least 5, then $c(G) \geq \delta(G)$.

Proof: Let $G$ have girth 5 with $\delta(G) = d$, and suppose $d-1$ cops are playing.
Let $C$ be the set of vertices in $G$ occupied by cops, and suppose by way of contradiction that $C$ is a dominating set.
Let $u$ be a vertex outside of $C$. Then $N(u) = X \cup Y$, where $X \subset C$ and $Y \subset V(G) - C$.
Note $X$ and $Y$ partition $N(u)$, so $|X| + |Y| \geq d$. 
Theorem (Aigner and Fromme, 1984)

Recall C is a dominating set, so each vertex in Y is adjacent to some vertex in C. Since G has girth 5, then for all \( y \in Y, x \in X \), y is nonadjacent to x. And no two vertices of Y are adjacent to the same vertex in C. So each vertex of Y is adjacent to a unique vertex of \( C \setminus X \).

Then \( d - 1 = |C| \geq |X| + |Y| \geq d \), a contradiction.
Theorem (Aigner and Fromme, 1984)

So $C$ is not a dominating set, hence there is some vertex adjacent to $u$ which is not adjacent to any vertex in $C$, and the robber may move there in round 0.

Suppose in round $t \geq 0$, that the robber occupies some vertex $u_t$ such that no vertex adjacent to $u_t$ is in $C$.

By induction, suppose such a vertex exists for $t-1$. 
Theorem (Aigner and Fromme, 1984)

So at time $t$, the robber is on a vertex $u_{t-1}$ nonadjacent to all vertices in $C$. The girth of $G$ is 5, so each cop is adjacent to at most one neighbor of $u_{t-1}$. And $d(u_{t-1}) \geq d > |C|$, so the robber may move somewhere not adjacent to any cop. This may continue indefinitely, and therefore $c(G) \geq d = \delta(G)$.

Corollary: $c(\text{Pete}) \geq 3$, hence $c(\text{Pete}) = 3$. 

Recall, for some functions $f$ and $g$ with fixed domain,

$f = O(g)$ if there exist positive constants $a, c$ such that $f(n) \geq cg(n)$ for all $n \geq a$.

Also, $f = o(g)$ if $\lim_{x \to \infty} |f(x)/g(x)| = 0$.

So if $f = o(1)$, then $\lim_{x \to \infty} |f(x)| = 0$. 
Upper bounds

● Currently the best known upper bound is

\[ c(G) = O \left( \frac{n}{2^{(1-o(1))} \sqrt{\log_2 n}} \right). \]

● This was proved by Lu and Peng in 2011.

● Meyniel's Conjecture from 1985 states

\[ c(n) = O(n^{1/2}) \]

● This is one of the biggest open problems on the cop number.
Retracts

- Let $H = G - v$ for some vertex $v$ in $G$.
- $H$ is a retract of $G$ if there is a homomorphism $f$ from $G$ to $H$ such that $f(x) = x$ for all $x \in V(H)$. Recall that a homomorphism preserves edges, i.e. if $xy \in E(G)$ then $f(x)f(y) \in E(H)$.
- If $H$ is a retract of $G$, then $c(H) \leq c(G)$.

Proof: Let $c(G) = k$ for some $k$. We will play two games, one in $G$ and one in $H$ such that when $C$ moves from vertex $u$ to $v$ in $G$, then $f(C)$ moves from $f(u)$ to $f(v)$ in $H$. 
Retracts

Let the cops play in G with R restricted to H. G is k-cop-win, so at some point the cops will be about to win in G. Then R and each of its neighbors in H (and its neighbors in G - H = v) are adjacent to some cop.

Under the retraction, the edge RC becomes Rf(C), and vC becomes vf(C). So $N[R] \subseteq N[f(C)]$ in H, and the robber loses in H in the next round. So $c(H) \leq k$.

Corollary: A retract of a cop-win graph is cop-win
Corners

A **corner** is a vertex \( u \) in \( G \) such that for some vertex \( v \) in \( G \), \( N[u] \subseteq N[v] \).

Lemma: If \( G \) is a cop-win graph, then \( G \) contains at least one corner.

Proof: Consider the next to last move of the robber in a cop-win graph \( G \). The robber can move anywhere in \( N[R] \), but the cop gets him on the next round. So \( C \leftrightarrow N[R] \), hence \( N[R] \subseteq N[C] \).

Lemma: \( G-u \) is a retract of \( G \).
Theorem

In a cop-win graph $G$ with $n \geq 5$ vertices, the cop can capture the robber in at most $n-3$ rounds.

Proof: base case: $n=5$. Trust me, it works.

induction step: By induction, suppose the theorem holds for some $n \geq 5$, and let $G$ be a cop-win graph on $n+1$ vertices.

$G$ contains a corner $u$ dominated by some vertex $v$ ($N[u] \subseteq N[v]$). So $G-u$ is a retract of $G$, hence $G-u$ is cop-win on $n$ vertices.
Theorem

By induction, the cop can win on G-u in at most n-3 rounds.
The cop plays her winning strategy in G-u and captures the shadow of R.
So if R is on u, C plays as if R is on v.
After n-3 rounds, either the robber has been captured, or R is on u and C is on v.
So the robber is caught in at most (n+1)-3 moves in G.
If $k$ is fixed, then determining if $c(G) \leq k$ can be done in polynomial time.
If $k$ is not fixed, this problem is NP-hard.
It's unknown if the cop number problem is even in NP.
Goldstein and Reingold conjectured in 1995 that the cop number problem is EXPTIME-complete.
Problems in exponential time have running time $O(2^{p(n)})$, where $p(n)$ is a polynomial of $n$. 