Domination in Graphs

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Final Project in Graph Theory, 2012
Outline

1 Intro
   - History
   - Example

2 Domination
   - Definitions
   - Examples

3 Domination Bounds
   - Theorems
   - More definitions
   - Examples
   - Lemma and Theorem
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History

- 1960 - The study of domination in graphs began.
- 1862 - C.F. De Jaenisch attempted to determine the minimum number of queens required to cover an $n \times n$ chess board.
- **N-Queens Problem** You can place $n$ queens on an $n \times n$ chessboard so that no two queens attack each other. A solution exists for all natural numbers $n$ except 2 and 3.
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Can you place 8 queens on the 8x8 board so that no two queens attack each other?
There is currently no known formula for the exact number of solutions.
Counting Solutions

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Note that the six queens puzzle has fewer solutions than the five queens puzzle.
1892 - W.W. Rouse Ball reported three basic types of problems that chess players studied during this time.

1. **Covering**: Determine the minimum number of chess pieces of a given type that are necessary to cover (attack) every square of an $nxn$ chess board.

2. **Independent Covering**: Determine the smallest number of mutually nonattacking chess pieces of a given type that are necessary to dominate every square of an $nxn$ board.

3. **Independence**: Determine the maximum number of chess pieces of a given type that can be placed on an $nxn$ chess board such that no two pieces attack each other.
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1958 - Claude Berge introduced the domination number of a graph.

In a graph $G$, a set $S \subseteq V(G)$ is a dominating set if every vertex not in $S$ has a neighbor in $S$.

The domination number $\gamma(G)$ is the minimum size of a dominating set in $G$. 

Definitions
Petersen Graph

Determine the domination number of the Petersen graph.
The Petersen graph has domination number 3. Each vertex dominates itself and three others, so at least three vertices are needed. Since the graph has diameter 2, the neighbors of a single vertex form a dominating set.
Example with Cycles and Paths

Determine the domination number of $C_n$ and $P_n$ for the following graphs.
$\gamma(C_n) = \gamma(P_n) = \lceil n/3 \rceil$. $C_n$ and $P_n$ both have a maximum degree of 2, and therefore each vertex can dominate itself and up to 2 other vertices. Picking every third vertex starting from the second, and using the last when $n$ is not divisible by 3, yields a dominating set of size $\lceil n/3 \rceil$. 
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Bounds in terms of different graph parameters

- A vertex of degree \( k \) dominates itself and \( k \) other vertices; thus every dominating set in a \( k \)-regular graph \( G \) has size at least \( n(G)/(k + 1) \).
- The set of all the vertices is a dominating set in any graph; therefore \( \gamma(G) \leq n(G) \).
Every $n$-vertex graph with minimum degree $k$ has a dominating set of size at most $n \frac{1+\ln(k+1)}{k+1}$. $\gamma(G) \leq n(G)$.

**Proof** Let $G$ be a graph with minimum degree $k$. Given $S \subseteq V(G)$, let $U$ be the set of vertices not dominated by $S$.

We claim that some vertex $y$ outside of $S$ dominates at least $\|U\|(k+1)/n$ vertices of $U$. Each vertex in $U$ has at least $k$ neighbors, so $\sum_{v \in U} \|N[v]\| \geq \|U\|(k+1)$.

Each vertex of $G$ is counted at most $n$ times by these $\|U\|$ sets, so some vertex $y$ appears at least $\|U\|(k+1)/n$ times and satisfies the claim.
proof continued ...

- We iteratively select a vertex that dominates the most of the remaining undominated vertices.
- We have proved that when \( r \) undominated vertices remain, after the next selection at most \( r \left(1 - \frac{(k + 1)}{n}\right) \) undominated vertices remain.
- Hence after \( n \left(\frac{\ln(k+1)}{k+1}\right) \) steps the number of undominated vertices is at most
  \[
  n \left(1 - \frac{k+1}{n}\right) n \frac{\ln(k+1)}{(k+1)} < n e^{-\ln(k+1)} = \frac{n}{k+1}.
  \]
- The selected vertices and these remaining undominated vertices together form a dominating set of size at most
  \[
  n \left(1 + \ln(k+1)\right) \frac{1}{k+1}.
  \]
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A dominating set $S$ in $G$ is a **connected dominating set** if $G[S]$ is connected, an **independent dominating set** if $G[S]$ is independent, and a **total dominating set** if $G[S]$ has no isolated vertex.
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Peterson Graph

- Determine the minimum size of a total dominating set in the Petersen graph.
The Peterson graph has total domination number 4. A total dominating set $S$ must include a neighbor of every vertex in $S$. Hence $S$ must contain two adjacent vertices. This pair leaves four undominated vertices. Adding a neighbor of the original pair dominates at most three of these, since the graph is 3-regular. One vertex and its neighbors form a total dominating set of size four.
The dominating set, connected dominating set, independent dominating set, and total dominating set of $C_6$, $C_7$, $P_6$, and $P_7$. 

![Graphs of $C_6$, $C_7$, $P_6$, and $P_7$ with dominating sets highlighted.](image-url)
Queen Problem

Find the total domination number of the Queens problem on a 8x8 board.
Queen Solution

- The total domination number of the Queens problem on a 8x8 board is 5.
Queen Solution to 15x15 board

- 15x15 board with domination number 5
There is currently no known formula for the total domination number for any $nxn$ board.

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**Lemma**: A set of vertices in a graph is an independent dominating set if and only if it is a maximal independent set.

**Proof**: Among independent sets, $S$ is maximal if and only if every vertex outside $S$ has a neighbor in $S$, which is the condition for $S$ to be a dominating set.
**Theorem:** Every claw-free graph has an independent dominating set of size $\gamma(G)$.

**Proof** Let $S$ be a minimum dominating set in a claw-free graph $G$. Let $S'$ be a maximal independent subset of $S$. Let $T = V(G) - N(S')$. Let $T'$ be a maximal independent subset of $S$.

- Since $T'$ contains no neighbor of $S'$, $S' \cup T'$ is independent.
- Since $S'$ is maximal in $S$, we have $S \subseteq N(S')$.
- Since $T'$ is maximal in $T$, $T'$ dominates $T$. 

![Diagram of graph elements]
proof continued ...

- It remains to show that $|S' \cup T'| \leq \gamma(G)$.
- Since $S'$ is maximal in $S$, $T'$ is independent, and $G$ is claw-free, each vertex of $S - S'$ has at most one neighbor in $T'$.
- Since $S$ is dominating, each vertex of $T'$ has at least one neighbor in $S - S'$.
- Hence $|T'| \leq |S - S'|$, which yields $|S' \cup T'| \leq |S| = \gamma(G)$.  

![Diagram showing sets $N(S')$, $S$, $S'$, $T'$, and $T$]
1968 - Vadim G. Vizing stated his conjecture that for any graphs $G$ and $H$, $\gamma(G \Box H) \geq \gamma(G)\gamma(H)$. Vizing’s conjecture is perhaps the biggest open problem in the field of domination theory in graphs.

Vizing’s Conjecture has been proven true for certain classes of graphs

1) In 1979 Barcalkin and German established a class of graphs known as BG-graphs for which Vizing’s conjecture holds.

2) A corollary of this result is that Vizing’s conjecture holds for all graphs with domination number equal to 2, graphs with domination number equal to 2-packing number.

3) The result that Vizing’s conjecture is true for trees was also proved by Faudree, Schelp, Shreve, Chen, and Piotrowski.
Vizing’s Conjecture has also been proven true for these classes of graphs

4) In 1995 Hartnell and Rall established Vizing’s conjecture for a larger class of graphs called Type $\chi$ which is an extension of BG-graphs.

5) Most recent related Vizing’s conjecture was by Bresar and Rall in 2009, which defined the fair domination number of a graph $G$. 
Sources I

- Douglas B. West
  Introduction to Graph Theory 2nd Ed.

- Jennifer M. Tarr
  Domination in Graphs

  Queens Graphs for chessboards on the Torus
Summary Continued