What are Splines?

Definition:
A function made up of piece-wise polynomial functions that maintains a high degree of smoothness at the points of contact (known as knots).
Splines were originally pieces of thin malleable metal or wood used to draw smooth curves.

The control points were pieces of metal or wood, called ducks, that were used to bend and hold the spline in place to draw the curve.

Especially during World War II, splines were used to make the most capable vessels.
The main application of Bézier curves is in computer graphics such as animation.
The main application of Bézier curves is in computer graphics such as animation and fonts.
Given control points, splines navigate throughout them creating a smooth curve.

Most often splines are used for interpolation or approximation with curves that exists only in 2D or 3D.
Basic definitions and equations

Definitions:
- Interpolation: evaluation with a line that passes through all points.
- Approximation: evaluation with a line that passes through only the first and last points.

General Bézier Curve equation

\[ P(t) = \sum_{i=1}^{n} \binom{n}{i} (1 - t)^{n-i} t^i w_i \]

Cubic Bézier curve

\[ P(t) = (1 - t)^3 + 3t(1 - t)^2 + 3t^2(1 - t) + t^3 \]
Cubic Bézier curve

\[ P(t) = (1 - t)^3 + 3t(1 - t)^2 + 3t^2(1 - t) + t^3 \]

- Let \( P_0 \) be defined as \( t = 0 \) and \( P_3 \) as \( t = 1 \).

Cubic Bézier Curves are awesome!

Using interpolation for the first and last points (\( P_0 \) and \( P_3 \)) and approximation for the second and third points (\( P_1 \) and \( P_2 \)), we get a cubic polynomial.
Cubic Bézier curve

\[ P(t) = (1 - t)^3 + 3t(1 - t)^2 + 3t^2(1 - t) + t^3 \]

- Let \( P_0 \) be defined as \( t = 0 \) and \( P_3 \) as \( t = 1 \).

Cubic Bézier Curves are awesome!

Using interpolation for the first and last points (\( P_0 \) and \( P_3 \)) and approximation for the second and third points (\( P_1 \) and \( P_2 \)), we get a cubic polynomial.

- The Bernstein polynomials are used to weight the influence of each control point in the set.
- The Bernstein polynomials define the basis for the Cubic Bézier Curve
- Coined by Sergei Natanovich Bernstein in 1912 but became more relevant in 1962 when Pierre Bézier applied them to his cubic curves.
Bernstein polynomials

- The Bernstein polynomials define the basis for the Cubic Bézier Curve
- Coined by Sergei Natanovich Bernstein in 1912 but became more relevant in 1962 when Pierre Bézier applied them to his cubic curves.

The Bernstein Basis for the Cubic Bézier Curve

\{(1 − t)^3, 3t(1 − t)^2, 3t^2(1 − t), t^3\}

The weight influences how fast and how far the curve moves toward \(P_1\) and \(P_2\) from \(P_0\) to \(P_3\)
### Properties of Bernstein Polynomials

- **Symmetry:** For \( k = 0, 1, 2, 3 \),
  \[ B_{3,k}(1 - t) = B_{3,3-k}(t) \]
- **Basis:** \( B_{3,0}(t), B_{3,1}(t), B_{3,2}(t), B_{3,3}(t) \) form a basis for the vector space of polynomials of degree at most three (the standard/canonical basis is \( 1, t, t^2, t^3 \)).
- **Partition of Unity:** The set of functions (in this case the Bernstein polynomials) sum to one for all values of \( t \).

### Properties of the Cubic Bézier Curve

- **Degree:** \( P(t) \) has degree at most 3.
- **Symmetry:** \( P(1 - t) \) is the same as the Bézier curve with control points in the opposite order: \( P_3, P_2, P_1, P_0 \).
- **Convex hull:** For \( 0 \leq t \leq 1 \), the Bézier curve lies entirely in the convex hull of its control points.
Importance of Properties

- Convex hull:

- Check of Symmetry: $B_{3,k}(1 - t) = B_{3,3-k}(t)$
- Proof of Partition of Unity:
The standard basis in coordinate space is called the canonical basis and for a cubic is $\{1, t, t^2, t^3\}$.

We can change the basis using a matrix to go from the canonical basis to the Bernstein Basis.

The reason that the Bernstein polynomials can be used is because they are a linear combination of basis polynomials.

Also looking at the basis as a linear combination allows us to see they are in fact linearly independent and form a basis.
Use the inverse matrix to transform the Bernstein Basis to the canonical Basis.

\[
\begin{bmatrix}
B_1(t) \\
B_2(t) \\
B_3(t) \\
B_4(t)
\end{bmatrix} =
\begin{bmatrix}
1 & -3 & 3 & -1 \\
0 & 3 & -6 & 3 \\
0 & 0 & 3 & -3 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 \\
t \\
t^2 \\
t^3
\end{bmatrix}
\]
Proof of linear independence

We know that the power basis spans the space of polynomials and is linear independent. So by showing a linear combination of the Bernstein polynomials can be written as a power basis we will have proven its linear independence.

- Assume the Bernstein polynomials are linearly independent
- Recall:

\[ 0 = c_0 B_0, n(t) + c_1 B_1, n(t) + \ldots + c_n B_n, n(t), \]

where all \( c_1, c_2, \ldots, c_n = 0, \)
Thank you, from yours truly!

The Berenstain Bears
Bibliography

- Bernstein Polynomials (Kenneth. I Joy)

- Splines and B-splines, an Introduction

- Cubic Bezier Curves
  - http://www.math.ucla.edu/~baker/149.1.02w/handouts/bb\textsubscript{bez}ier.pdf