

Free Triangle Orders

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Abstract. We define a new class of ordered sets, called free triangle orders. These are ordered sets represented by a left-to-right ordering on geometric objects contained in a horizontal strip in the plane. The objects are called “free triangles,” and have one vertex on each of the two boundaries of the strip and one vertex in its interior. These ordered sets generalize the classes of trapezoid and triangle orders studied in (Bogart et al., 1998; Ryan, 1998; Ryan, 1999), represented by trapezoids and triangles respectively, contained within a strip in the plane, and are a special case of the orders of tube dimension 2 introduced by (Habib et al., 1992), which are represented by any set of convex bodies contained within a strip in the plane. Our main result is that the class of free triangle orders strictly contains the class of trapezoid orders.

Keywords: trapezoid order, triangle order, free triangle order

Mathematics Subject Classifications (2000): 06A06, 05C62

1. Introduction

Suppose that R is a finite set of closed intervals on the real line (drawn horizontally). If x and y are intervals in R , then we define $x < y$ if and only if x lies entirely to the left of y . We call this order relation the *standard ordering* of R . An ordered set P is called an *interval order* if there exists an ordered set R , whose elements are intervals ordered by the standard ordering, such that $P \cong R$. This isomorphism is called an *interval representation* of P . We also refer to the set R as an interval representation of P .

Note that we may think of interval orders as ordered sets that are “representable” using a set of one-dimensional geometric objects. We now generalize to two dimensions. Suppose we are given two horizontal lines in the plane, B_1 and B_2 , with B_1 below B_2 , which we call the *baselines* of our representation, and a finite set R of trapezoids, each of which has one base on each of the baselines. We allow trapezoids to have one or two bases of length zero. If x and y are trapezoids in R , we define $x < y$ if and only if x and y are disjoint, and the base of x on B_1 is to the left of the base of y on B_1 . We call this order relation the *standard ordering* of R . An ordered set P is called a *trapezoid order* if

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there exists such an ordered set of trapezoids R such that $P \cong R$. This isomorphism is called a *trapezoid representation* of P .

If R is instead a finite set of triangles, with a vertex on one baseline and a base on the other, then this isomorphism is called a *triangle representation* of P , and P is called a *triangle order*. Note that different triangles in R may have bases on different baselines.

If R is a finite set of triangles with one vertex on B_1 , one vertex on B_2 , and one vertex strictly between B_1 and B_2 , then this isomorphism is called a *free triangle representation* of P , and P is called a *free triangle order*. We call the two vertices of each triangle on B_1 and B_2 the *bound vertices* of the triangle, and the third vertex its *free vertex*.

The focus of this paper is on containment relations between these classes of ordered sets. The following two lemmas are previously known results in this direction. Lemma 1.1 is implied by work in (Ryan, 1999), but never explicitly stated. We provide a proof here for completeness.

Lemma 1.1. *Every interval order is a triangle order, but not conversely.*

Proof. Suppose P is an interval order, and R is an interval representation of P . We consider the line containing R as B_1 , and draw B_2 above it. For each interval in R , we place a vertex on B_2 above its center, and connect the two endpoints of the interval to this new vertex to form a triangle. The new set of triangles is a triangle representation of P .

Conversely, the ordered set $\mathbf{2} + \mathbf{2}$ shown in Figure 1 is not an interval order (Trotter, 1992). However, Figure 1 shows a triangle representation of $\mathbf{2} + \mathbf{2}$. \square

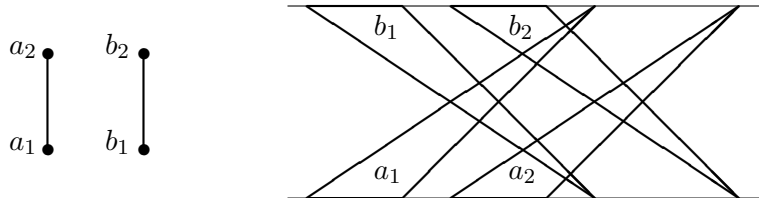


Figure 1. The ordered set $\mathbf{2} + \mathbf{2}$ and a triangle representation of this ordered set

Theorem 1.2. (Ryan, 1999) *Every triangle order is a trapezoid order, but not conversely.*

As one might suspect, the extra freedom of the free vertices in a free triangle representation allows us to represent a strictly larger class of ordered sets.

Lemma 1.3. *Every triangle order is a free triangle order, but not conversely.*

Proof. To show that every triangle order is a free triangle order, we may simply shift a vertex of every triangle in a triangle representation slightly off a baseline to form a free triangle representation. Since our orders are finite, we may do this without changing the order relation on the triangles.

We put off the second part of the proof until Section 5, where we exhibit an example of a free triangle order that is not a trapezoid order. In particular, this will prove the “not conversely” part of the statement above. \square

The relationship between trapezoid orders and free triangle orders is the main focus of this paper.

Trapezoid, triangle, and free triangle orders are all examples of ordered sets with tube dimension 2. An ordered set with tube dimension 2 is an ordered set representable with convex bodies contained between two baselines in \mathbb{R}^2 , and intersecting both baselines (Habib et al., 1992). Habib, Kelly, and Möhring proved that being an ordered set with tube dimension n is a comparability invariant for positive integers n , so the results in this paper hold for the corresponding classes of interval graphs, trapezoid graphs, triangle graphs, and free triangle graphs as well.

2. Notation

For a trapezoid t in a trapezoid representation R , for $k = 1$ and 2 , we denote the two vertices of t on B_k by $v_k^-(t)$ and $v_k^+(t)$, meaning the left and right vertices of t on B_k , respectively. We denote the horizontal coordinates of $v_k^-(t)$ and $v_k^+(t)$ by $x_k^-(t)$ and $x_k^+(t)$. We denote the left and right edges of t by $e^-(t)$ and $e^+(t)$, respectively.

If R is a triangle or free triangle representation of some ordered set, and t is a triangle in R which has only one vertex on the baseline B_k , we denote this vertex by $v_k(t)$, and the horizontal coordinate of $v_k(t)$ by $x_k(t)$. Additionally, if t is a free triangle, we denote the free vertex of t by $v_f(t)$.

We call a triangle t in a free triangle representation *left-pointing* (respectively, *right-pointing*) if the free vertex $v_f(t)$ of t is less than (respectively, greater than) the edge of t between $v_1(t)$ and $v_2(t)$. Every triangle in a free triangle representation is therefore either left-pointing or right-pointing.

3. Pleasant Representations

We will call a trapezoid representation R *pleasant* if it satisfies the following properties.

1. All bases of trapezoids in R , on both baselines, have positive length.
2. All vertices of all trapezoids in R are distinct.
3. All vertices of all trapezoids in R have a horizontal coordinate in \mathbb{N} .
4. B_1 is the line $y = 0$, and B_2 is the line $y = 1$.
5. $x_2^+(t) < x_1^-(t)$ for every trapezoid t in R .

Lemma 3.1. *If P is a trapezoid order, then there exists a pleasant trapezoid representation of P .*

Proof. Let R be a trapezoid representation of P . We perform a series of transformations on R that convert R into a pleasant trapezoid representation of P . Note that this proof relies heavily on the fact that R is a finite set.

First, we replace any base of length zero in R with a base of some small positive length. Since there are a finite number of trapezoids in R , we may choose this length small enough so that relations between trapezoids in R are unchanged. Similarly, if two or more vertices coincide, we may shift one or more of them slightly to the left or right so that they are distinct. The trapezoid representation R now satisfies Properties 1 and 2.

Now that every vertex is distinct, the vertices on each baseline are linearly ordered. Therefore we can find an order-preserving function f from these vertices to the natural numbers. We place each vertex v at the point on its baseline with horizontal coordinate $f(v)$. We now perform an affine transformation on R that translates the baselines B_1 and B_2 to the lines $y = 0$ and $y = 1$. Once again, neither of these transformations alter the relations between trapezoids in R , and R now additionally satisfies Properties 3 and 4.

Finally, we slide B_1 to the right as much as necessary (in other words, add a constant amount to the horizontal coordinate of each vertex on B_1) so that R additionally satisfies Property 5. The trapezoid representation R is now a pleasant trapezoid representation of P as desired. \square

4. The Main Result

The main result of this paper is that every trapezoid order is also a free triangle order. We will first need to prove a lemma about a particular structure that occurs in trapezoid representations that turns out to be essential to our argument.

We define a (*right-facing*) *asterisk* in a trapezoid representation to be an ordered triple of three trapezoids (p, q, r) , such that $x_2^+(p) < x_2^+(q) < x_2^+(r)$ and $x_1^+(r) < x_1^+(q) < x_1^+(p)$, as shown in Figure 2. We will not need to use the analogous definition of left-facing asterisks.



Figure 2. An asterisk in a trapezoid representation

Note that the horizontal edges of the trapezoids in Figure 2 all lie along the two baselines and are drawn off of them for clarity. This will be true of later figures as well.

Lemma 4.1. *Every trapezoid order P has a trapezoid representation that satisfies the following property:*

Property A. *For every asterisk (p, q, r) in R , $e^+(q)$ passes to the left of the intersection of $e^+(p)$ and $e^+(r)$.*

Proof. Let S be a pleasant trapezoid representation of P . We will convert S into a pleasant trapezoid representation with Property A.

We apply the function $\phi(x, y) = (3^x, y)$ to S to obtain a pleasant trapezoid representation R of P . Since S is pleasant, every vertex of every trapezoid in S has a horizontal coordinate in \mathbb{N} . Therefore if the horizontal coordinate of one vertex v is greater than that of another vertex w in S , the horizontal coordinate of v is greater than twice that of w in R . We call this the *doubling property* of ϕ .

We claim that R has Property A. Consider an asterisk (p, q, r) in R . Since ϕ is order-preserving on horizontal coordinates, (p, q, r) is an asterisk in R if and only if it is an asterisk in S . Since S is pleasant, we have $x_2^+(r) < x_1^+(r)$ in S . Combining this with the definition of an asterisk gives us $x_2^+(p) < x_2^+(q) < x_2^+(r) < x_1^+(r) < x_1^+(q) < x_1^+(p)$ in

S . Since we applied ϕ to S to obtain R , by the doubling property, each of these numbers is less than half of the next in R . For ease of reference, we will denote these horizontal coordinates in R by the numbers a, b, c, d, e , and f , as in Figure 3. So we have $2a < b, 2b < c, \dots, 2e < f$.

We denote the intersection of $e^+(p)$ and $e^+(q)$ by (w_1, w_2) and the intersection of $e^+(p)$ and $e^+(r)$ by (z_1, z_2) , also shown in Figure 3. We want to show that the line segment $e^+(q)$ passes to the left of the point (z_1, z_2) , as it appears in the figure.

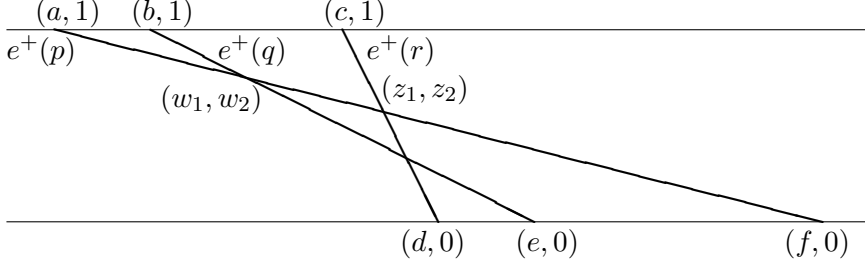


Figure 3. The right edges of an asterisk

We claim it suffices to show that $w_1 < z_1$ in R . Suppose this were true. Since $a < b$ and $e < f$, the slope of $e^+(q)$ is more negative than the slope of $e^+(p)$. Therefore on any horizontal line less than w_2 , $e^+(q)$ is to the left of $e^+(p)$. Since the slope of $e^+(p)$ is negative, z_2 is less than w_2 . Therefore $e^+(q)$ passes to the left of (z_1, z_2) .

We now prove that $w_1 < z_1$. The line segment $e^+(p)$ connects the points $(a, 1)$ and $(f, 0)$, so its equation is $y = \frac{1}{a-f}(x - f)$. The line segment $e^+(q)$ connects the points $(b, 1)$ and $(e, 0)$, so its equation is $y = \frac{1}{b-e}(x - e)$. Therefore solving $\frac{1}{a-f}(x - f) = \frac{1}{b-e}(x - e)$ gives us

$$w_1 = \frac{e(f - a) - f(e - b)}{(f - a) - (e - b)}. \quad (1)$$

Similarly,

$$z_1 = \frac{d(f - a) - f(d - c)}{(f - a) - (d - c)}. \quad (2)$$

To prove the lemma, we show that the right-hand side of Equation (1) is less than the right-hand side of Equation (2).

Since $0 < a < b < c < d < e < f$, the expressions $(f - a) - (e - b)$ and $(f - a) - (d - c)$ are both positive. If we multiply through by both of these expressions and cancel like terms, we see that the desired inequality $\frac{e(f-a)-f(e-b)}{(f-a)-(e-b)} < \frac{d(f-a)-f(d-c)}{(f-a)-(d-c)}$ is equivalent to $b(f - d) -$

$c(f - e) < a(e - d)$. Since $a(e - d)$ is positive, it suffices to show that $b(f - d) < c(f - e)$.

To prove this last inequality, note that the doubling property yields $2e < f$, and hence $f < 2(f - e)$. Also, $d > 0$ yields $f - d < f$, so $f - d < 2(f - e)$. Finally, the doubling property yields $2b < c$. We conclude that $b(f - d) < 2b(f - e) < c(f - e)$, which completes the proof. \square

Now we are ready to prove the theorem.

Theorem 4.2. *If P is a trapezoid order then P is a free triangle order.*

Proof. We will in fact prove a stronger statement, namely that P has a free triangle representation in which every triangle is left-pointing. For this proof, we define the standard ordering on a set of polygons, some of which are trapezoids and some of which are free triangles, analogously to the standard orderings defined in Section 1.

Let R be a pleasant trapezoid representation of P that has Property A. Assume that R has m trapezoids, labelled $1, \dots, m$, in any order. We convert these trapezoids into free triangles in this order; let R_p be the standard order on the resulting set of polygons after the trapezoids labelled $1, \dots, p$ have been converted.

We perform the following procedure to form R_p from R_{p-1} if, and only if, it is possible to do so in such a way that the polygon p overlaps exactly the same polygons in R_p as in R_{p-1} . We convert the trapezoid p into a triangle by removing one of the two vertices $v_2^-(p)$ or $v_1^-(p)$ and replacing p with the convex hull of its remaining three vertices, as in Figure 4. We can then move the remaining left vertex of p off of its baseline by some small amount, as in Lemma 1.3, to convert p into a left-pointing free triangle, without changing relations between the polygons in R_p .

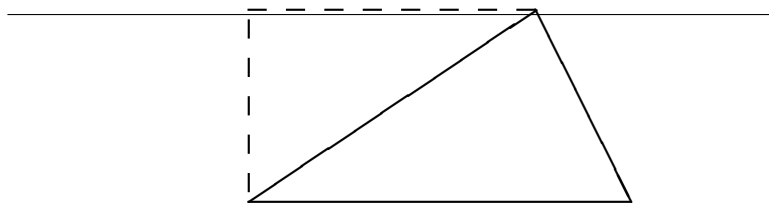


Figure 4. Removing $v_2^-(p)$

If it is not possible to convert p into a left-pointing free triangle in this way then there is at least one polygon that is incomparable to p but would be less than p upon the removal of $v_2^-(p)$, and at least one polygon that is incomparable to p but would be less than p upon the removal of $v_1^-(p)$. In this case we will remove both $v_1^-(p)$ and $v_2^-(p)$ and replace them with a single free vertex.

We form R'_p by replacing p by the edge $e^+(p)$ from R_{p-1} . Note that $R'_p \not\cong R_{p-1}$ since in R'_p , p is to the right of a set of elements T formerly overlapping p in R_{p-1} . We will form R_p by placing the third vertex of p to transform p into a left-pointing free triangle.

Denote the set of elements of R'_p incomparable to p by $I(p)$. Our goal is to place the third vertex $v_f(p)$ of p in such a way that we add all the polygons of T to $I(p)$, and no other polygons. Note that all of the polygons in R_{p-1} , whether trapezoids or triangles, retain their original right edge from R . Therefore, $v_f(p)$ must be placed to the left of $e^+(t)$ for all t in T . In other words, if $E^+(t)$ is the set of all points to the left of $e^+(t)$, then $v_f(p)$ must be placed to the left of the right boundary \mathcal{R} of the set $\bigcap_{t \in T} E^+(t)$. The curve \mathcal{R} is made up of segments of the right edges $e^+(t)$ of polygons t in T .

The set T consists of polygons of three types:

- Type 1.** Polygons that would become less than p in R_{p-1} upon the removal of $v_1^-(p)$ only.
- Type 2.** Polygons that would become less than p in R_{p-1} upon the removal of $v_2^-(p)$ only.
- Type 3.** Polygons that would become less than p in R_{p-1} only upon the removal of both $v_1^-(p)$ and $v_2^-(p)$.

Since there exists at least one polygon of type 1 and at least one polygon of type 2 in T , \mathcal{R} is less than p on both B_1 and B_2 . Therefore \mathcal{R} must have a segment of the right edge of a polygon of type 1 as its bottom-most piece, and a segment of the right edge of a polygon of type 2 as its top-most piece.

Therefore there exist two polygons t_1 and t_2 , of different types among $\{1, 2, 3\}$, such that the point at which the edges $e^+(t_1)$ and $e^+(t_2)$ cross is on \mathcal{R} . We call this point x , and we place $v_f(p)$ just to the left of x to create R_p . This placement puts $v_f(p)$ less than \mathcal{R} , so p overlaps every polygon in T in R_p .

The only difference in the orderings on R'_p and R_p is that more elements have been added to $I(p)$. We have seen that every polygon in T has been added to $I(p)$. It remains for us to show that no other polygons have been added. There are two cases to consider.

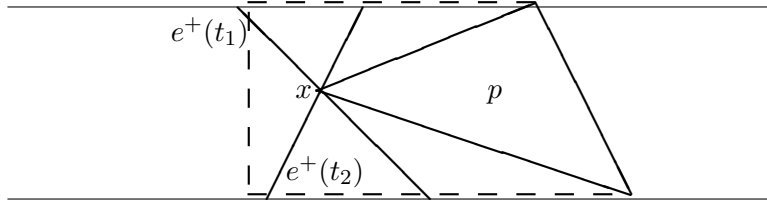


Figure 5. Placing the free vertex of p , case 1

Case 1. One of the polygons t_1 or t_2 has type 3. This case is shown in Figure 5. In this case, x must be contained in p in R_{p-1} . Therefore p gets strictly smaller in the conversion from R_{p-1} to R_p , and it cannot possibly overlap polygons in R_p that it did not in R_{p-1} .

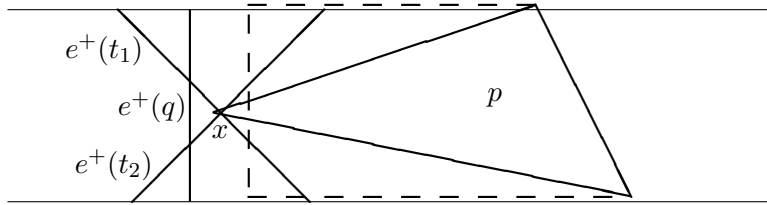


Figure 6. Placing the free vertex of p , case 2

Case 2. The polygon t_1 has type 1, and the polygon t_2 has type 2. This case is shown in Figure 6. In this case, we argue by contradiction. Suppose that there exists a polygon q such that q is left of p in R_{p-1} , and q overlaps p in R_p (this is not true in the figure). This means that as a trapezoid, p is to the right of $e^+(q)$, but as a free triangle, p overlaps $e^+(q)$. In other words, $x_1^+(q) < x_1^+(t_1)$, $x_2^+(q) < x_2^+(t_2)$, and $e^+(q)$ crosses $e^+(t_1)$ and $e^+(t_2)$. Then we must have $x_2^+(t_1) < x_2^+(q) < x_2^+(t_2)$ and $x_1^+(t_2) < x_1^+(q) < x_1^+(t_1)$. Therefore (t_1, q, t_2) is an asterisk in R_p . But $e^+(t_1)$, $e^+(q)$, and $e^+(t_2)$ are the same edges in R_p that they were in R . Therefore (t_1, q, t_2) is an asterisk in R . Hence by Lemma 4.1, since R has Property A, $e^+(q)$ is less than x . Therefore $q < p$ in R_p , as it appears in Figure 6.

Therefore R_{p-1} and R_p are isomorphic ordered sets, and so R_m is a free triangle representation of P . Therefore P is a free triangle order. \square

5. Separating Examples

In this section we show that not every free triangle order is a trapezoid order, and not every ordered set is a free triangle order. Recall that the ordered set S_n , the standard example of an n -dimensional ordered set, contains the 1-element and $(n - 1)$ -element subsets of $[n]$, ordered by inclusion (Trotter, 1992). Figure 7 shows the ordered set S_3 .

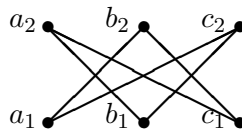


Figure 7. The ordered set S_3

In what follows, we distinguish the intersection of sets in the plane from the intersection of ordered sets common in the theory of ordered sets by calling the latter *order intersection*. The *interval-order dimension* of an ordered set P is the minimal number of interval orders whose order intersection is P .

Lemma 5.1. (Trotter, 1992) *The ordered set S_n has interval-order dimension n .*

Lemma 5.2. *An ordered set P is a trapezoid order if and only if it has interval-order dimension at most 2.*

Proof. If R is a trapezoid representation of P , the ordered sets $R \cap B_1$ and $R \cap B_2$ are interval orders whose order intersection is P . Conversely, if I_1 and I_2 are interval orders whose order intersection is P , we form a trapezoid representation of P by placing the intervals in I_1 along B_1 , placing the intervals in I_2 along B_2 , and adding left and right edges to each pair of intervals to form trapezoids. \square

Theorem 5.3. *The ordered set S_3 is a free triangle order and not a trapezoid order.*

Proof. Figure 8 shows a free triangle representation of S_3 . However, S_3 is not a trapezoid order by the above two lemmas. \square

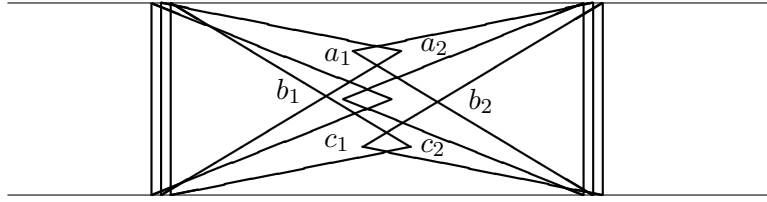


Figure 8. A free triangle representation of S_3

Theorem 5.4. *The ordered set S_3^+ shown in Figure 9 is not a free triangle order.*

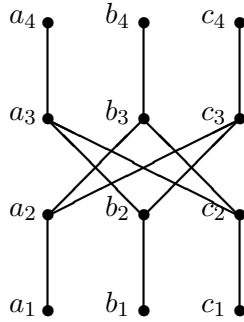


Figure 9. The ordered set S_3^+

Proof. By way of contradiction, let R be a free triangle representation of S_3^+ . We label each triangle in R by its corresponding element in S_3^+ . As in Lemma 3.1, we may assume that the bound vertices of the triangles in R are all distinct. We claim that $x_i(a_3) < x_i(a_2)$ for $i = 1$ or 2 (but not necessarily both). Suppose to the contrary that $x_1(a_2) < x_1(a_3)$ and $x_2(a_2) < x_2(a_3)$. Let $e(t)$ be the edge of the free triangle t between $v_1(t)$ and $v_2(t)$. Since the triangle a_1 is to the left of the triangle a_2 , it follows that a_1 is to the left of $e(a_2)$. Similarly, a_4 is to the right of $e(a_3)$.

Since $x_1(a_2) < x_1(a_3)$ and $x_2(a_2) < x_2(a_3)$, $e(a_2)$ is to the left of $e(a_3)$, which implies that a_1 is to the left of a_4 . But this is a contradiction, since $a_1 \parallel a_4$. Similar arguments apply for the pairs b_2 and b_3 , and c_2 and c_3 . Therefore

$$x_i(a_3) < x_i(a_2) \text{ for } i = 1 \text{ or } 2, \quad (3)$$

$$x_i(b_3) < x_i(b_2) \text{ for } i = 1 \text{ or } 2, \text{ and} \quad (4)$$

$$x_i(c_3) < x_i(c_2) \text{ for } i = 1 \text{ or } 2. \quad (5)$$

However, it is not possible that both $x_1(a_3) < x_1(a_2)$ and $x_1(b_3) < x_1(b_2)$, by the following reasoning. Since $b_2 < a_3$ and $a_2 < b_3$, $x_1(b_2) < x_1(a_3)$ and $x_1(a_2) < x_1(b_3)$. Therefore if both $x_1(a_3) < x_1(a_2)$ and $x_1(b_3) < x_1(b_2)$, then $x_1(b_2) < x_1(a_3) < x_1(a_2) < x_1(b_3) < x_1(b_2)$, which is impossible. By similar reasoning, we cannot satisfy any pair of the equations (3), (4), and (5) for the same value of i . Since i can only take on two values, R cannot exist. \square

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