Math 399 – Topics in Graph Theory
Definitions List

Graphs

A **graph** G is a set of **vertices** (singular: **vertex**), denoted $V(G)$, and a set of **edges**, which are two-element subsets of $V(G)$, denoted $E(G)$.

Two vertices x and y in G are **adjacent**, denoted $x \sim y$, if there is an edge containing them.

Suppose G and H are two graphs. An **isomorphism** ϕ from G to H is a one-to-one, onto function that maps vertices of G to vertices of H, such that if x and y are vertices in G, then $x \sim y$ if and only if $\phi(x) \sim \phi(y)$. Two graphs are said to be **isomorphic** if there exists an isomorphism between them.

A **walk** in a graph is a sequence of vertices v_1, v_2, ..., v_k, such that $v_i \sim v_{i+1}$ for all $1 \leq i \leq k-1$.

A **path** is a walk with no repeated vertex.

A **cycle** is a walk with no repeated vertex except that $v_1 = v_k$, and for which $k > 3$.

A graph G is called **connected** if, for every pair of vertices in G, there exists a path between them.

A **tree** is a connected graph with no cycles.

The **degree** of a vertex v, denoted $d(v)$, is the number of edges containing v.

A **leaf** in a tree is a vertex of degree 1.

A **Hamiltonian cycle** is a cycle which contains every vertex in the graph. A graph is **Hamiltonian** if it contains a Hamiltonian cycle.

The **product** of two graphs G and H, denoted $G \times H$, is the graph with vertex set $\{(x, y) | x \in G, y \in H\}$ and with an edge between (x_1, y_1) and (x_2, y_2) if and only if either ($x_1 = x_2$ and $y_1 \sim y_2$), or ($y_1 = y_2$ and $x_1 \sim x_2$).

A **clique** in a graph G is a subset of the vertices of G which are all adjacent to one another.

An **independent set** is a subset of the vertices of G, none of which are adjacent to one another.

The **Ramsey number** $R(n, k)$ is the smallest number such that every graph on $R(n, k)$ vertices has either a clique of size n or an independent set of size k.

Directed Graphs

A **directed graph** G is a set of vertices and a set of edges, which are ordered pairs of the vertices of G.

A **tournament** T is a directed graph in which for every pair of vertices x and y, T contains exactly one of the two possible edges between them. If the edge (x, y) is in a tournament, we say that x **dominates** y.

The **indegree** $d_i(x)$ of a vertex x in a directed graph G is the number of vertices that dominate x. The **outdegree** $d_o(x)$ of x is the number of vertices dominated by x.

A directed graph G is called **transitive** if, whenever (x, y) and (y, z) are edges in G, then (x, z) is an edge in G.

A **directed path** in a directed graph G is a sequence of distinct vertices in G such that each vertex dominates the next. A **Hamiltonian path** in a directed graph G is a directed path that contains every vertex of G.

A **directed cycle** is a directed path for which the first and last vertices are the same. A **Hamiltonian cycle** in a directed graph G is a directed cycle that contains every vertex in G.

A directed graph G is called **strongly connected** if, for every pair of vertices x and y of G, there is a directed path from x to y.

Ordered Sets

An **ordered set** or **partially ordered set** or **poset** P is a transitive directed graph in which for every pair of vertices x and y, P contains at most one of the two possible edges between them.

We call the vertices of an ordered set P the **elements** of P. We call the edges of P the **relations** of P, and we write $x < y$ if (x, y) is an edge in P.

If $x < y$ in P and there is no element z in P such that $x < z < y$, then we say that y **covers** x.

In an **order diagram** of an ordered set, if $x < y$, we draw x below y on the page, and draw an upward-slanting (undirected) edge from x up to y if y covers x.

An **isomorphism** ϕ from an ordered set P to an ordered set Q is a one-to-one, onto function that maps elements of P to elements of Q, such that if x and y are elements of P, then $x < y$ if and only if $\phi(x) < \phi(y)$. Two ordered sets are said to be **isomorphic** if there exists an isomorphism between them.

Two elements x and y in an ordered set P are said to be **comparable** if either $x < y$ or $y < x$ in P, otherwise x and y are **incomparable** (sometimes written $x \parallel y$).

A **linear order** or **total order** or **chain** is an ordered set in which every pair of elements are comparable.

An **antichain** is an ordered set in which every pair of elements are incomparable.

An ordered set Q is called an **extension** of an ordered set P if the elements of P and Q are the same, and the set of relations of P is a subset of the set of relations of Q. In other words, if $x < y$ in P, then $x < y$ in Q, but not necessarily conversely. Q is called a **linear extension** of P if Q is an extension of P, and also a linear order.

The **intersection** of two ordered sets P and Q with the same elements is the ordered set with the same elements as P and Q, and whose relation set is the intersection of the relation set of P and the relation set of Q. In other words, $x < y$ in $P \cap Q$ if and only if $x < y$ in both P and Q.

The **dimension** $\dim(P)$ of an ordered set P is the minimum number of linear orders whose intersection is P.