

Math 399 – Topics in Graph Theory

Graph Domination Handout

A *dominating set* of a graph G is a set of vertices \mathcal{D} such that, for every vertex x of G , either $x \in \mathcal{D}$, or there exists a vertex $y \in \mathcal{D}$ such that $x \sim y$. The *domination number* $\gamma(G)$ of G is the smallest cardinality of a dominating set of G .

1. Find the domination number of P_n , the path with n vertices.
2. A set of vertices \mathcal{D} in G is said to be a *minimal* dominating set if the removal of any vertex from \mathcal{D} makes it no longer dominating. Prove that a set of vertices \mathcal{D} in a connected graph G is a minimal dominating set if and only if, for every vertex $x \in \mathcal{D}$, one of the following properties holds:
 - P1.** There exists a vertex $y \notin \mathcal{D}$ such that $x \sim y$ and y is adjacent to no other vertex in \mathcal{D} .
 - P2.** The vertex x is adjacent to no other vertex in \mathcal{D} .
3. Prove that if \mathcal{D} is a minimal dominating set of a connected graph G , then $V(G) - \mathcal{D}$ is a dominating set of G .
4. If G is a connected graph with n vertices, prove that $\gamma(G) \leq \frac{n}{2}$.
5. Find three connected graphs on 6 vertices with $\gamma(G) = 1, 2,$ and 3 , respectively. This proves that the bound in the previous problem is best possible.
6. A graph G is called *γ -critical* if the removal of any vertex of G decreases $\gamma(G)$. Find a connected γ -critical graph G with $\gamma(G) = k$ for each positive integer k .
7. Prove that no tree is γ -critical.
8. The *bondage number* $b(G)$ of a graph G is the smallest number of edges whose removal increases $\gamma(G)$. What are the possibilities for the bondage number of a tree?