Math 399 – Topics in Graph Theory
Introduction to Graph Theory Handout

A *graph* G is a set of *vertices* (singular: *vertex*), denoted $V(G)$, and a set of *edges*, which are two-element subsets of $V(G)$, denoted $E(G)$. Two vertices x and y in G are *adjacent*, denoted $x \sim y$, if there is an edge containing them. It is usually easier to visualize a graph by drawing the vertices as points in the plane, and the edges as line segments or curves connecting pairs of vertices. For example, the graph with vertex set $\{1, 2, 3, 4\}$ and edge set $\{\{1, 2\}, \{2, 3\}\}$ is shown below.

However, this graph can be drawn in many different ways. Here is another drawing of the same graph.

It is usually easier, when we talk about a graph G, to talk about a particular drawing of G rather than list the edges as sets. However, do not forget that a graph is not the same as a drawing of a graph. Also note that since $E(G)$ is a set, and each edge is a set, we do not allow more than one edge between the same two vertices, or any edge between a vertex and itself.

1. What is the maximum number of edges possible in a graph with n vertices?
2. What is the minimum number of edges possible in a graph with n vertices?
3. How many graphs are there with vertex set $\{1, 2, 3, 4, 5\}$?

Suppose G and H are two graphs. An *isomorphism* ϕ from G to H is a one-to-one, onto function that maps vertices of G to vertices of H, such that if x and y are vertices in G, then $x \sim y$ if and only if $\phi(x) \sim \phi(y)$. Two graphs are said to be *isomorphic* if there exists an isomorphism between them. The word isomorphic comes from the Latin *iso*, meaning same, and *morph*, meaning shape or form. In other words, if two graphs are isomorphic, they really are the same graph, the only difference being that they might have different labels. We usually group isomorphic graphs together and think of them as the same graph. Formally, being isomorphic is an equivalence relation, which divides the set of graphs up into equivalence classes. When we refer to "a graph," often we really mean an equivalence class of isomorphic graphs. In this case, we do not need to label the vertices of our graphs, since the labels don't really matter.

4. Which of the graphs shown below are isomorphic? The graph on the left is called the *Petersen graph.*

5. Describe all of the distinct, non-isomorphic graphs with 5 vertices. How many are there?

A *walk* in a graph is a sequence of vertices v_1, v_2, ..., v_k, such that $v_i \sim v_{i+1}$ for all $1 \leq i \leq k-1$. A *path* is a walk with no repeated vertex. A *cycle* is a walk with no repeated vertex except that $v_1 = v_k$, and for which $k > 3$.

6. A graph G is called *connected* if, for every pair of vertices in G, there exists a path between them. Which of the graphs you found in problem 5 are connected?

7. A *tree* is a connected graph with no cycles. Which of the graphs you found in problem 5 are trees?

8. State and prove a theorem about the relationship between the number of vertices and the number of edges in a tree.