Math 399 – Topics in Graph Theory
Ordered Sets Handout

An *ordered set* or *partially ordered set* or *poset* P is a transitive directed graph in which for every pair of vertices x and y, P contains at most one of the two possible edges between them (compare this with the definition of a transitive tournament from last week). We will restrict ourselves to finite ordered sets in this handout, although most of what follows is true in some form for infinite ordered sets as well. We call the vertices of an ordered set P the *elements* of P. We call the edges of P the *relations* of P, and we write $x < y$ if (x, y) is an edge in P. If $x < y$ in P and there is no element z in P such that $x < z < y$, then we say that y *covers* x.

We can draw an ordered set as a directed graph, or using an *order diagram*, which is usually much easier to understand. In an order diagram, if $x < y$, we draw x below y on the page, and draw an upward-slanting (undirected) edge from x up to y if y covers x. Therefore, by the transitive property, $x < y$ if and only if there is an upward-slanting path from x to y in the order diagram. Below is shown the ordered set with elements a, b, c, and d, and relations $a < b$, $b < c$, $a < c$, and $a < d$.

1. (a) Verify that any set of positive integers, with $x < y$ if and only if $x < y$ in the usual sense, is an ordered set.
   (b) Verify that any set of positive integers, with $x < y$ if and only if $x|y$ and $x \neq y$, is an ordered set. These ordered sets are said to be "ordered by divisibility."
   (c) Verify that any set of finite sets, with $X < Y$ if and only if $X \subseteq Y$ and $X \neq Y$, is an ordered set. These ordered sets are said to be "ordered by inclusion."
2. An *isomorphism* ϕ from P to Q is a one-to-one, onto function that maps elements of P to elements of Q, such that if x and y are elements of P, then $x < y$ if and only if $\phi(x) < \phi(y)$ (compare this with the definition of an isomorphism between graphs from the first week). Two ordered sets are said to be *isomorphic* if there exists an isomorphism between them. Prove that the set of divisors of the number 30, ordered by divisibility, is isomorphic to the set of subsets of $\{1, 2, 3\}$, ordered by inclusion.
3. Two elements x and y in an ordered set P are said to be *comparable* if either $x < y$ or $y < x$ in P, otherwise x and y are *incomparable* (sometimes written $x \parallel y$). A *linear order* or *total order* or *chain* is an ordered set in which every pair of elements are comparable (in directed graph terms, a transitive tournament). Prove that every linear order with n elements is isomorphic to the set $\{1, 2, \ldots, n\}$, ordered by the usual $<$.

4. An *antichain* is an ordered set in which every pair of elements are incomparable. An ordered set Q is called an *extension* of an ordered set P if the elements of P and Q are the same, and the set of relations of P is a subset of the set of relations of Q. In other words, if $x < y$ in P, then $x < y$ in Q, but not necessarily conversely. Q is called a *linear extension* of P if Q is an extension of P, and also a linear order. How many linear extensions are there of an antichain with n elements?

5. Prove that if x and y are incomparable elements of an ordered set P, then there exists a linear extension of P in which $x < y$.

6. The *intersection* of two ordered sets P and Q with the same elements is the ordered set with the same elements as P and Q, and whose relation set is the intersection of the relation set of P and the relation set of Q. In other words, $x < y$ in $P \cap Q$ if and only if $x < y$ in both P and Q. Prove that if P is an ordered set, then P is the intersection of all linear extensions of P.

7. The *dimension* $\dim(P)$ of an ordered set P is the minimum number of linear orders whose intersection is P. By the previous problem, all (finite) ordered sets have finite dimension. Find the dimension of the ordered set in problem 2.

8. Find a 6-element ordered set with dimension 3.

9. Find a $2n$-element ordered set with dimension n. It turns out that this is the least number of elements of an ordered set with dimension n.

10. Prove that if P is an ordered set, then $\dim(P) \leq n$ if and only if P is isomorphic to some set of n-tuples of positive integers, ordered coordinate-wise (in other words, $(x_1, x_2, \ldots, x_n) < (y_1, y_2, \ldots, y_n)$ if and only if $x_i < y_i$ for all i). This is the intuitive reason for using the word "dimension." By the remark in problem 7, this also proves that every (finite) ordered set is isomorphic to some set of n-tuples of positive integers, ordered coordinate-wise.