

Math 399 – Topics in Graph Theory

Ramsey Theory Handout

A *clique* in a graph G is a subset of the vertices of G which are all adjacent to one another. An *independent set* is a subset of the vertices of G , none of which are adjacent to one another. The *Ramsey number* $R(n, k)$ is the smallest number such that every graph on $R(n, k)$ vertices has either a clique of size n or an independent set of size k . Ramsey's theorem says that the Ramsey number $R(n, k)$ exists (i.e. is a finite number) for all positive integers n and k . Frank Ramsey proved this very influential theorem in 1930 at the age of 26, and died that same year.

1. Find $R(2, k)$ for all positive integers k .
2. Show that $R(n, k) = R(k, n)$ for all positive integers n and k .
3. Find $R(3, 3)$.
4. Show that $R(n, k) \leq R(n, k - 1) + R(n - 1, k)$.
5. The *degree* of a vertex v , denoted $d(v)$, is the number of edges containing v . Show that

$$\sum_{v \in V(G)} d(v) = 2|E(V)|.$$

One graph theorist has called this the fundamental theorem of graph theory.

6. Show that if $R(n, k - 1)$ and $R(n - 1, k)$ are both even, then $R(n, k) < R(n, k - 1) + R(n - 1, k)$.
7. Use the formulas from problems 4 and 6 to obtain upper bounds on $R(3, 4)$, $R(4, 4)$, and $R(3, 5)$.
8. Prove that $R(3, 4) = 9$.
9. To this day, $R(5, 5)$ is not known. Paul Erdős, possibly the greatest mathematician of the 20th century, said,

“Aliens invade the earth and threaten to obliterate it in a year's time unless human beings can find the Ramsey number $[R(5, 5)]$. We could marshal the world's best minds and

fastest computers, and within a year we could probably calculate the value. If the aliens demanded the Ramsey number $[R(6, 6)]$, however, we would have no choice but to launch a preemptive attack.”

Here’s why Erdős said this. It is known that $43 \leq R(5, 5) \leq 49$. Suppose that you wrote a computer program to check whether $R(5, 5) = 43$. You might conceivably have to check every 5-vertex subset of every 43-vertex graph to search for cliques and independent sets. How many 5-vertex subsets is this? How long would it take a computer to check these, at the rate of 20 billion subsets a second?

10. Show that $\binom{n+k-2}{n-1} = \binom{n+k-3}{n-1} + \binom{n+k-3}{n-2}$ for all n and k greater than 1.
11. Show that $R(n, k) \leq \binom{n+k-2}{n-1}$.
12. What fraction of graphs on v vertices have cliques of size n ? Use this to show that $2^{n/2} < R(n, n)$. You may need the inequality $\binom{n}{k} < \frac{n^k}{k!}$. This is called the *probabilistic method* of proof in graph theory.