Math 399 – Topics in Graph Theory
Tournaments Handout

A *directed graph* G is a set of vertices and a set of edges, which are ordered pairs of the vertices of G (compare this definition with the definition of a graph from the first week). If x and y are two vertices of G, we draw the directed edge (x, y) as an arrow with its tail at x and its head at y. Note that in a directed graph, there are two possible edges between x and y, the edge (x, y), and the edge (y, x). A *tournament* T is a directed graph in which for every pair of vertices x and y, T contains exactly one of the two possible edges between them (think of a round-robin tournament, so that the vertices represent teams, and the edge (x, y) represents the fact that team x beat team y). If the edge (x, y) is in a tournament, we say that x *dominates* y.

1. How many directed graphs are there with vertex set $\{1, 2, 3, 4, 5\}$?
2. How many tournaments are there with vertex set $\{1, 2, 3, 4, 5\}$?
3. The *indegree* $d_i(x)$ of a vertex x in a directed graph G is the number of vertices that dominate x. The *outdegree* $d_o(x)$ of x is the number of vertices dominated by x. Prove that
$$\sum_{x \in V(G)} d_i(x) = \sum_{x \in V(G)} d_o(x).$$
4. A *directed path* in a directed graph G is a sequence of distinct vertices in G such that each vertex dominates the next. A *Hamiltonian path* in G is a directed path that contains every vertex of G. Prove that every tournament contains a Hamiltonian path.
5. A directed graph G is called *transitive* if, whenever (x, y) and (y, z) are edges in G, then (x, z) is an edge in G. Prove that if a tournament T is transitive, then T has a "champion," i.e., a vertex that dominates every other vertex in T.
6. Prove that a tournament is transitive if and only if it contains exactly one Hamiltonian path.
7. A *directed cycle* is a directed path for which the first and last vertices are the same. Prove that a tournament T is transitive if and only if it contains no directed cycles.
8. A directed graph G is called *strongly connected* if, for every pair of vertices x and y of G, there is a directed path from x to y. Prove that no transitive tournament is strongly connected.
9. A *Hamiltonian cycle* is a directed cycle that contains every vertex in the graph. Prove that every strongly connected tournament contains a Hamiltonian cycle.