

# Math 386: Graph Theory

## Spring 2012

### Definitions List

- A **graph**  $G$  is a set of vertices  $V(G)$ , a set of edges  $E(G)$ , and a relation that associates with each edge two vertices called its **endpoints**.
- A **loop** is an edge whose endpoints are the same.
- **Multiple edges** are edges with the same endpoints.
- A **simple graph** is a graph with no loops or multiple edges.
- Two vertices  $v$  and  $w$  in  $G$  are **adjacent**, denoted  $v \leftrightarrow w$ , if there is an edge between them.
- If the vertex  $v$  is an endpoint of the edge  $e$ , then  $e$  and  $v$  are **incident**.
- The **degree**  $d(v)$  of a vertex  $v$  is the number of edges incident to it, counting loops twice.
- An **isomorphism** between two simple graphs  $G$  and  $H$  is a bijection  $\varphi : V(G) \rightarrow V(H)$  such that  $vw \in E(G)$  if and only if  $\varphi(v)\varphi(w) \in E(H)$ .
- Two graphs are **isomorphic** if there exists an isomorphism between them.
- The **complement** of a simple graph  $G$  is the graph  $\overline{G}$  with  $V(G) = V(\overline{G})$ , and  $uv \in \overline{G}$  if and only if  $uv \notin G$ .
- The **path**  $P_n$  is the graph with  $n$  vertices  $v_1, \dots, v_n$  and edges  $v_1v_2, \dots, v_{n-1}v_n$ .
- The **cycle**  $C_n$  is the graph with  $n$  vertices  $v_1, \dots, v_n$  and edges  $v_1v_2, \dots, v_{n-1}v_n, v_nv_1$ .
- The **complete graph**  $K_n$  is the graph with  $n$  vertices in which every pair of vertices are adjacent.
- A graph  $G$  is **bipartite** if its vertices can be partitioned into two sets  $X$  and  $Y$ , called **partite sets**, such that every edge in  $E(G)$  is between a vertex in  $X$  and a vertex in  $Y$ .
- The **complete bipartite graph**  $K_{m,n}$  is the bipartite graph with partite sets of  $m$  vertices and  $n$  vertices, such that every pair of vertices in different sets are adjacent.
- The **adjacency matrix**  $A$  of a graph  $G$  with  $n$  vertices is the  $n \times n$  matrix with rows and columns indexed by the vertices of  $G$ , where the number in the  $i$ th row and  $j$ th column of  $A$  is the number of edges between the  $i$ th and  $j$ th vertex, counting loops twice.

- The **incidence matrix**  $M$  of a graph  $G$  with  $n$  vertices and  $e$  edges is the  $n \times e$  matrix with rows indexed by the vertices of  $G$  and columns indexed by the edges of  $G$ , where the number in the  $i$ th row and  $j$ th column of  $A$  is the number of times vertex  $i$  and edge  $j$  are incident, counting loops twice.
- A **walk** in a graph is an alternating sequence of vertices and edges  $v_1e_1v_2e_2\dots v_ke_k$ , such that each term of the sequence is incident to the next.
- A **trail** is a walk with no repeated edge.
- A **path** is a walk with no repeated vertex.
- A **circuit** is a trail whose first and last vertices are the same.
- A **cycle** is a circuit with no repeated vertex other than the first and last vertex.
- The **length** of a walk, trail, path, circuit, or cycle in a graph is the number of edges in it (counting repeated edges multiple times).
- The **girth** of a graph  $G$  is the length of its shortest cycle. If  $G$  has no cycles, then its girth is infinite.
- A graph  $G$  is **connected** if, for every pair of vertices in  $G$ , there exists a path between them.
- A **component** of a graph  $G$  is a maximal connected subgraph of  $G$ .
- A **cut edge** or **cut vertex** of a graph  $G$  is an edge or vertex whose deletion increases the number of components of  $G$ .
- A **subgraph**  $H$  of a graph  $G$  is a graph such that  $V(H) \subseteq V(G)$  and  $E(H) \subseteq E(G)$ .
- An **induced subgraph**  $H$  of a graph  $G$  is a subgraph of  $G$  obtained by deleting a set of vertices.
- An **Eulerian circuit** (alternatively, **Euler tour**) of a graph  $G$  is a circuit which contains every edge of  $G$ .
- A graph is **Eulerian** if it contains an Eulerian circuit.
- A **Hamiltonian cycle** of a graph  $G$  is a cycle which contains every vertex of  $G$ .
- A graph is **Hamiltonian** if it contains a Hamiltonian cycle.
- A graph  $G$  is **regular** if every vertex of  $G$  has the same degree. If that degree is  $k$ , then  $G$  is  **$k$ -regular**.
- The **neighborhood** of a vertex  $v$  is the set of vertices adjacent to  $v$ .
- The **degree sequence** of a graph is the list of degrees of the graph, in non-increasing order.

- A **tree** is a connected graph with no cycles.
- A **leaf** of a tree is a vertex of degree 1.
- A **spanning subgraph**  $H$  of a graph  $G$  is a subgraph of  $G$  such that  $V(H) = V(G)$ .
- A **spanning tree** of a graph  $G$  is a spanning subgraph of  $G$  which is a tree.
- The **distance**  $d(v, w)$  between two vertices  $v$  and  $w$  is the length of the shortest path between them.
- The **diameter** of a graph  $G$  is the largest distance between any pair of vertices. If  $G$  is disconnected, then its diameter is infinite.
- A **forest** is a graph with no cycles.
- A **directed graph** or **digraph**  $G$  is a set of vertices  $V(G)$ , a set of edges  $E(G)$ , and a relation that associates with each edge an ordered pair of vertices called its endpoints. The first vertex in the ordered pair is the **tail** and the second vertex is the **head** of the edge.
- If  $G$  is a directed graph, the **underlying graph** of  $G$  is the graph obtained from  $G$  by un-ordering each of its edges.
- If  $G$  is a graph, an **orientation** of  $G$  is obtained from  $G$  by ordering each of its edges.
- A **tournament** is an orientation of a complete graph.
- The **indegree**  $d^-(v)$  of a vertex  $v$  is the number of edges with head  $v$ . The **outdegree**  $d^+(v)$  of a vertex  $v$  is the number of edges with tail  $v$ .
- A **directed walk** in a graph is an alternating sequence of vertices and edges  $v_1e_1v_2e_2\dots v_ke_k$ , such that  $e_i = v_iv_{i+1}$  for all  $1 \leq i \leq k - 1$ .
- A directed graph  $G$  is **strongly connected** if there is a directed path from every vertex to every other vertex in  $G$ .
- A directed graph is **weakly connected** if its underlying graph is connected.
- A directed graph is **acyclic** if it contains no directed cycles.
- A directed graph  $G$  is **transitive** if  $(x, y), (y, z) \in E(G)$  implies  $(x, z) \in E(G)$  for all  $x, y, z \in V(G)$ .
- A **matching** in a graph is a set of edges with no shared endpoints.
- A matching is **perfect** if its edges are incident to every vertex in  $G$ .
- An **independent set** of a graph  $G$  is a set of pairwise non-adjacent vertices.
- The **independence number**  $\alpha(G)$  is the maximum size of an independent set in  $G$ .

- A **vertex cover** of  $G$  is a set of vertices  $S$  such that every edge of  $G$  is incident to some vertex in  $S$ .
- An **edge cover** of  $G$  is a set of edges  $S$  such that every vertex of  $G$  is incident to some edge in  $S$ .
- A **vertex cut** of a graph  $G$  is a set of vertices  $S$  of  $G$  such that  $G - S$  is disconnected.
- A graph is  **$k$ -connected** if it has no vertex cut with fewer than  $k$  vertices.
- A graph has **connectivity**  $k$ , denoted  $\kappa(G)$ , if it is  $k$ -connected and not  $(k + 1)$ -connected.
- A **block** of a graph  $G$  is a maximal connected subgraph of  $G$  with no cut vertices.
- An **edge cut** of a graph  $G$  is a set of edges  $S$  of  $G$  such that  $G - S$  is disconnected.
- A graph is  **$k$ -edge-connected** if it has no edge cut with fewer than  $k$  edges.
- A graph has **edge-connectivity**  $k$ , denoted  $\kappa'(G)$ , if it is  $k$ -edge-connected and not  $(k + 1)$ -edge-connected.