Math 486: Graph Theory
Fall 2008
Definitions List

- A ***graph*** G is a set of vertices $V(G)$, a set of edges $E(G)$, and a relation that associates with each edge two vertices called its ***endpoints***.
- A ***loop*** is an edge whose endpoints are the same.
- ***Multiple edges*** are edges with the same endpoints.
- A ***simple graph*** is a graph with no loops or multiple edges.
- Two vertices v and w in G are ***adjacent***, denoted $v \leftrightarrow w$, if there is an edge between them.
- If the vertex v is an endpoint of the edge e, then e and v are ***incident***.
- The ***degree*** $d(v)$ of a vertex v is the number of edges incident to it.
- An ***isomorphism*** between two simple graphs G and H is a bijection $\varphi : V(G) \to V(H)$ such that $vw \in E(G)$ if and only if $\varphi(v)\varphi(w) \in E(H)$.
- Two graphs are ***isomorphic*** if there exists an isomorphism between them.
- The ***complement*** of a simple graph G is the graph $\overline{G}$ with $V(G) = V(\overline{G})$, and $uv \in \overline{G}$ if and only if $uv \notin G$.
- The ***path*** P_n is the graph with n vertices $v_1, \ldots, v_n$ and edges $v_1v_2, \ldots, v_{n-1}v_n$.
- The ***cycle*** C_n is the graph with n vertices $v_1, \ldots, v_n$ and edges $v_1v_2, \ldots, v_{n-1}v_n$, v_nv_1.
- The ***complete graph*** K_n is the graph with n vertices in which every pair of vertices are adjacent.
- A graph G is ***bipartite*** if its vertices can be partitioned into two sets X and Y, such that every edge in $E(G)$ is from a vertex in X to a vertex in Y.
- The ***complete bipartite graph*** $K_{m,n}$ is the bipartite graph with parts of sizes m and n, such that every pair of vertices in different sets are adjacent.
- The ***adjacency matrix*** A of a graph G with n vertices is the $n \times n$ matrix with rows and columns indexed by the vertices of G, where the number in the ith row and jth column of A is the number of edges between the ith and jth vertex.
- The ***incidence matrix*** M of a graph G with n vertices and e edges is the $n \times e$ matrix with rows indexed by the vertices of G and columns indexed by the edges of G, where the number in the ith row and jth column of A is the number of times vertex i and edge j are incident.

- A ***walk*** in a graph is an alternating sequence of vertices and edges $v_1 e_1 v_2 e_2 ... v_k e_k$, such that each term of the sequence is incident to the next.
- A ***trail*** is a walk with no repeated edge.
- A ***path*** is a walk with no repeated vertex.
- A ***circuit*** is a trail whose first and last vertices are the same.
- A ***cycle*** is a circuit with no repeated vertex other than the first and last vertex.
- The ***length*** of a walk, trail, path, circuit, or cycle in a graph is the number of edges in it (counting repeated edges multiple times).
- The ***girth*** of a graph G is the length of its shortest cycle. If G has no cycles, then its girth is infinite.
- A graph G is ***connected*** if, for every pair of vertices in G, there exists a path between them.
- A ***component*** of a graph G is a maximal connected subgraph of G.
- A ***cut edge*** or ***cut vertex*** of a graph is an edge or vertex whose deletion increases the number of components.
- A ***subgraph*** H of a graph G is a graph such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.
- An ***induced subgraph*** H of a graph G is a subgraph of G obtained by deleting a set of vertices.
- An ***Eulerian circuit*** (alternatively, ***Euler tour***) of a graph G is a circuit which contains every edge of G.
- A graph is ***Eulerian*** if it contains an Eulerian circuit.
- A ***Hamiltonian cycle*** of a graph G is a cycle which contains every vertex of G.
- A graph is ***Hamiltonian*** if it contains a Hamiltonian cycle.
- A graph G is ***regular*** if every vertex of G has the same degree. If that degree is k, then G is k***-regular***.
- The ***neighborhood*** of a vertex v is the set of vertices adjacent to v.
- The ***degree sequence*** of a graph is the list of degrees of the graph, in non-increasing order.
- A ***tree*** is a connected graph with no cycles.
- A ***leaf*** of a tree is a vertex of degree 1.
- A ***spanning subgraph*** H of a graph G is a subgraph of G such that $V(H) = V(G)$.

- A ***spanning tree*** of a graph G is a spanning subgraph of G which is a tree.
- The ***distance*** $d(v, w)$ between two vertices v and w is the length of the shortest path between them.
- The ***diameter*** of a graph is the largest distance between any pair of vertices. If the graph is disconnected, then its diameter is infinite.
- A ***forest*** is a graph with no cycles.
- A function on the natural numbers $f(n)$ is ***order*** $g(n)$, denoted $O(g(n))$, if there exists positive constants c and a such that $f(n) \leq cg(n)$ for all $n \geq a$.
- The ***running time*** of an algorithm is the maximum number of steps it takes, as a function of the size of the input.
- A graph-related algorithm is ***good*** (alternatively, ***fast***, ***feasible***, or ***polynomial-time***) if, for any graph with n vertices, the algorithm has running time $O(n^k)$ for some positive integer k.
- A ***weighted graph*** is a graph with numbers, called ***weights***, assigned to each of its edges.
- In a weighted graph, the ***distance*** $d(v, w)$ between two vertices v and w is the minimum sum of the weights on the edges of a v, w-path.
- A k***-coloring*** of a graph G is an assignment of k colors to the vertices of G.
- A coloring is ***proper*** if adjacent vertices receive distinct colors.
- A graph G is k***-colorable*** if it has a proper k-coloring.
- The ***chromatic number*** $\chi(G)$ is the least k such that G is k-colorable.
- A graph G is ***color critical*** or k***-critical*** if $\chi(H) < \chi(G)$ for every proper subgraph H of G.
- An ***independent set*** of a graph G is a set of pairwise non-adjacent vertices.
- The ***independence number*** $\alpha(G)$ is the maximum size of an independent set in G.
- A ***clique*** in a graph G is a complete induced subgraph of G.
- The ***clique number*** $\omega(G)$ is the maximum size of a clique in G.
- A graph is k***-partite*** if its vertices can be partitioned into k independent sets.
- If G is a graph and xy is an edge of G, we ***subdivide*** the edge xy by adding the vertex z and replacing the edge xy by the two edges xz and zy.
- A ***subdivision*** of a graph G is a graph that can be obtained from G by a sequence of edge subdivisions.

- If G is a graph and $e = xy$ is an edge of G, we ***contract*** e, denoted by $G \cdot e$, by deleting e and identifying the vertices x and y.

- A ***minor*** of a graph G is a graph that can be obtained from G by a sequence of edge deletions and contractions.

- The ***join*** of two graphs G and H, denoted $G \vee H$, is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{xy \,|\, x \in V(G), y \in V(H)\}$.

- The ***Cartesian product*** of two graphs G and H, denoted $G \square H$, is the graph with vertex set $V(G) \times V(H)$ and edge set $\{(u,v)(u',v') \,|\, u = u' \text{ and } vv' \in E(H) \text{ or } v = v' \text{ and } uu' \in E(G)\}$.

- The ***chromatic polynomial*** $\chi(G;k)$ of the graph G is the function which takes any positive integer k and returns the number of proper k-colorings of G.

- A graph is ***planar*** if it can be drawn in the plane with no edge-crossings.

- A drawing of a planar graph in the plane with no edge-crossings is called a ***planar embedding*** of G or a ***plane graph***.

- A ***face*** of a plane graph G is a maximal connected subset of $\mathbb{R}^2 - G$.

- Two faces of a plane graph are ***adjacent*** if they share an edge.

- The ***dual*** G^* of a plane graph G is the graph with vertex set equal to the face set of G, and with an edge between two vertices for each edge between their corresponding faces.

- A ***directed graph*** or ***digraph*** G is a set of vertices $V(G)$, a set of edges $E(G)$, and a relation that associates with each edge an ordered pair of vertices called its endpoints. The first vertex in the ordered pair is the ***tail*** and the second vertex is the ***head*** of the edge.

- If G is a directed graph, the ***underlying graph*** of G is the graph obtained from G by un-ordering each of its edges.

- If G is a graph, an ***orientation*** of G is obtained from G by ordering each of its edges.

- A ***tournament*** is an orientation of a complete graph.

- The ***indegree*** $d^-(v)$ of a vertex v is the number of edges with head v. The ***outdegree*** $d^+(v)$ of a vertex v is the number of edges with tail v.

- A ***directed walk*** in a graph is an alternating sequence of vertices and edges $v_1 e_1 v_2 e_2 ... v_k e_k$, such that $e_i = v_i v_{i+1}$ for all $1 \leq i \leq k-1$.

- A directed graph G is ***strongly connected*** if there is a directed path from every vertex to every other vertex in G.

- A directed graph is ***weakly connected*** if its underlying graph is connected.
- A ***matching*** in a graph is a set of edges with no shared endpoints.
- A matching is ***perfect*** if its edges are incident to every vertex in G.
- A ***vertex cover*** of G is a set of vertices S such that every edge of G is incident to some vertex in S.
- An ***edge cover*** of G is a set of edges S such that every vertex of G is incident to some edge in S.
- A ***vertex cut*** of a graph G is a set of vertices S of G such that $G - S$ is disconnected.
- A graph is k***-connected*** if it has no vertex cut with fewer than k vertices.
- A graph has ***connectivity*** k, denoted $\kappa(G)$, if it is k-connected and not $(k+1)$-connected.
- A ***block*** of a graph G is a maximal connected subgraph of G with no cut vertices.
- An ***edge cut*** of a graph G is a set of edges S of G such that $G - S$ is disconnected.
- A graph is k***-edge-connected*** if it has no edge cut with fewer than k edges.
- A graph has ***edge-connectivity*** k, denoted $\kappa'(G)$, if it is k-edge-connected and not $(k+1)$-edge-connected.