Knot Theory
Summary of Sections 1.1-2.3

- **Definition:** A **knot** is a closed curve in space that does not intersect itself. A **ambient isotopy** is a deformation of a knot through space, such that at no time during the deformation does the knot intersect itself. Two closed curves are said to be the same knot (or *equivalent* knots) if and only if we can obtain one from the other via an ambient isotopy. (pages 2, 12)

- **Definition:** The **unknot** is the circle (and all of its deformations). (page 2)

- **Definitions:** Two other knots are the **trefoil** knot and the **figure eight** knot, shown on pages 2 and 3.

- **Definitions:** A **knot projection** is a projection of a knot into the plane, where at every place where the knot crosses itself, it is indicated which strand of the knot would be above which, if the knot were in three-dimensional space. A **planar isotopy** is a deformation of a knot projection *within the plane* that keeps every crossing intact. Two closed curves in the plane are said to be the same knot projection (or *equivalent* knot projections) if and only if we can obtain one from the other via a planar isotopy. (pages 3, 12)

- **Theorem:** Two equivalent knot projections are projections of two equivalent knots. However, equivalent knots may have inequivalent knot projections.

- **Definition:** A **crossing** of a knot projection is a point at which the knot crosses itself. Note that according to the definition, *knot projections* may have crossings, but *knots* do not. (page 3)

- **Definition:** A **strand** of a knot (or link) projection is a segment of curve that goes from one under-crossing to another, with only over-crossings in between. (page 23)

- **Definition:** An **alternating knot** is a knot with a projection in which, as you travel along the curve of the knot projection, you alternatively cross over and under other strands. Note that an alternating knot may have a projection that is not alternating. (page 7)

- **Definition:** The **composition** of two knots J and K, denoted $J\#K$, is formed in the following way. Take a knot projection of J and one of K, and put them next to each other. Find an arc of each of the two knots J and K on the "outside" of each projection. Delete a segment of arc from each, and connect the two arcs with two new arcs. (pages 7-8)

- **Definitions:** A knot is **composite** if it is the composition of two non-trivial knots. The knots J and K are **factor** knots of $J\#K$. A knot is **prime** if it is not composite. (pages 8-9)

- **Definitions:** An **orientation** of a knot is a chosen direction to travel around the knot. A knot together with an orientation is an **oriented** knot. Every knot has exactly two orientations. If a knot can be deformed back to itself, and in so doing, change its orientation, we say that the knot is **invertible**. (pages 10-11)

- **Theorem:** The composition $J\#K$ is unique if and only if one of the two knots J or K is invertible. (page 11)

- **Definition:** A knot is **amphicheiral** if it is equivalent to its mirror image. (page 15)

- **Definitions:** A **Reidemeister move** is one of three ways to change a projection of a knot, as shown on page 13. Note that the portion of the knot which is being changed must look *exactly* like the portion of the knot shown on page 13 to be considered a Reidemeister move.

- **Theorem:**

  1. Two knot projections that are related by a Reidemeister move are usually not equivalent.
  2. Two knot projections are projections of the same knot if and only if there is a sequence of Reidemeister moves that deforms one to the other.

- **Definitions:** A **link** is a set of closed curves in space that do not intersect themselves or each other. Everything said above for knots and knot projections holds true for links and link projections as well. The number of knots in a link is the number of **components** of the link. A link is **splittable** if its components are not knotted, i.e. if they can be deformed through an ambient isotopy to lie on different sides of a plane. (page 17)

- **Definition:** The **unlink of n components** is the link formed by n unknots, each of which can be split from each other. (page 18)

- **Definitions:** Three other links are the **Borromean rings**, the **Whitehead link**, and the **Hopf link**, shown on pages 17-18.

- **Definition:** The **linking number** of a link is computed as follows. Fix an orientation of each component of the link. For each crossing between components, count a +1 if a clockwise rotation of the under-strand lines it up with the over-strand, and a -1 otherwise. The sum of these numbers is the linking number of the oriented link. Note that we have only defined linking number for two-component links. (pages 18-19)

- **Definition:** A link is **Brunnian** if the link itself is non-trivial, but the removal of any one of its components leaves an unlink. (page 22)

- **Definition:** A knot projection or link projection is **tricolorable** if each of the strands in the projection can be colored one of three different colors, so that at each crossing, either three different colors appear or only one color appears, and at least two of the colors are used to color the knot projection or link projection. (page 23)

- **Definition:** The **stick number** $s(K)$ of a knot K is the least number of straight sticks necessary to make K. (page 29)
- **Definition:** The **crossing number** $c(K)$ of a knot K is the least number of crossings in any knot projection of K. (page 30)
- **Definition:** The **Dowker notation** of a knot projection is obtained in the following way. Starting at any under-strand of a crossing of the knot projection, label the crossings with the natural numbers, making any even number negative which corresponds to an under-strand and not an over-strand. When this process is complete, each crossing of the knot has two numbers, one even and one odd. The Dowker notation for the knot projection is the sequence of even numbers at the crossings labelled with the consecutive odd numbers. (pages 35-36)
- **RV's Lemma:** Form a loop in a knot by starting at a crossing and continuing in one direction until you return to the same crossing. Then you must pass through crossings an even number of times.
- **Theorem:** In the construction of the Dowker notation for a knot projection above, each crossing of the knot has exactly one even and one odd number labelling it.
- **Definition:** A **tangle** in a knot projection or link projection is a circular region of the plane, such that the knot or link crosses the circle exactly four times. Two tangles are considered to be equivalent if one can be obtained from the other by a sequence of Reidemeister moves in which the four crossings of the circle remain fixed. (page 41)
- **Definitions:** The ∞ tangle and the 0 tangle are the two tangles with no crossings, shown on page 42. An **integer tangle** is formed by starting with the 0 tangle and making some number of "left-handed twists." A positive integer tangle has over-strands which have positive slope. A negative integer tangle has over-strands which have negative slope.
- **Definition:** A **rational tangle** is formed from a sequence of integer tangles in the following way. We start with an integer tangle, and reflect it about the line through its northwest and southeast corners. Then we compose the new tangle with another integer tangle, as shown on page 43. We continue this process any number of times to obtain a rational tangle.
- **Definitions:** The **product** of two tangles T_1 and T_2 is formed by reflecting T_1 about the line through its northwest and southeast corners, then composing it with T_2. The **sum** of two tangles T_1 and T_2 is formed by simply composing them. A tangle obtained by a sequence of multiplications and additions of rational tangles is called an **algebraic tangle**. (page 48)
- **Definitions:** A **rational link** is formed from a rational tangle by connecting the two top strands and the two bottom strands of the tangle, as shown on page 46. An **algebraic link** is formed from an algebraic tangle by connecting the two top strands and the two bottom strands of the tangle. (pages 46, 48)

- **Definition:** One knot K is a **mutant** of another knot J if we can obtain K from J by removing a tangle from J, reflecting the tangle vertically or horizontally or rotating it 180 degrees, and replacing it. (page 49)

- **Definitions:** A **knot invariant** (or **link invariant**) is a property of a knot (or link) that is unchanged under ambient isotopy (equivalently, that is unchanged if we change knot projections of the knot). (page 21)

- **Theorem:** The following properties of knots (or links) are knot (or link) invariants.

  1. Being the unknot.
  2. Being an alternating knot.
  3. Being a prime knot.
  4. Being a composite knot.
  5. Being an invertible knot.
  6. Being an amphicheiral knot.
  7. Having n components.
  8. The linking number of a link.
  9. Being a Brunnian link.
  10. Being tricolorable.
  11. The stick number of a knot or link.
  12. The crossing number of a knot or link.
  13. The Dowker notation of a prime amphicheiral knot.
  14. Being a rational link.
  15. Being an algebraic link.
  16. The continued fraction of a rational link.

- **Open Question 1 [Josh]:** Is there an example of a knot projection which remains the same after a Reidemeister move of type III?

- **Open Question 2 [RV]:** Is there an example of an alternating knot K which has a non-alternating knot projection with $c(K)$ crossings?

- **Open Question 3 [RV]:** Prove that each Reidemeister move cannot be performed using a combination of the other two moves.