Systems of linear equations

Why systems of linear equations are cool:
Solutions to systems of linear equations can be thought of as solutions to matrix equations – solving the vector version of the equation 
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\[ A\mathbf{x} = \mathbf{b} \] (Sections 3.1, 3.2, 3.3).

Systems of linear equations describe linear objects in \( n \)-dimensional space: lines, planes, hyperplanes, etc. – in general called subspaces of \( \mathbb{R}^n \) (Sections 3.4 and 3.5).

Solutions to systems of linear equations can be thought of as inverse images of matrix functions: 
\[ A\mathbf{x} = \mathbf{b} \text{ asks for the set of all input vectors } \mathbf{x} \text{ that yield the output vector } \mathbf{b} \] (Sections 6.1–6.4).
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- Solutions to systems of linear equations can be thought of as inverse images of matrix functions: \( A\vec{x} = \vec{b} \) asks for the set of all input vectors \( \vec{x} \) that yield the output vector \( \vec{b} \) (Sections 6.1–6.4).
A **system of linear equations** is a set of equations of the form

\[
\begin{align*}
    a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
    a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_1 \\
    & \vdots \\
    a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_1.
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We write the system compactly using an augmented matrix:

\[
\begin{bmatrix}
  a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\
  a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{m1} & a_{m2} & \cdots & a_{mn} & b_m
\end{bmatrix}
\]
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- Multiply a row by a nonzero scalar.
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- Multiply a row by a nonzero scalar.
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- Add a multiple of one row to another.
A matrix is in **reduced row echelon form** if it satisfies the following properties:

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The location in the matrix corresponding to a leading 1 is called a *pivot position*, and the column it’s in is a *pivot column*.

The variables corresponding to the leading 1’s are called *leading variables*, and the others are *free variables*. 
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If we perform a series of row operations to get matrix $A$ into the matrix $U$ in reduced row echelon form, then we say $U$ is the **reduced row echelon form** of $A$. 
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**Theorem**

*Every matrix has exactly one reduced echelon form.*
A linear system is **homogenous** if all the $b$’s are 0.

Questions

1. How many solutions could a linear system have?

Theorem

Every linear system has either 0, 1, or $\infty$ solutions.

The **rank** of a matrix $A$ is the number of leading 1’s / pivot columns / leading variables in its reduced row echelon form.

Suppose a linear system has $m$ equations, $n$ unknowns, and rank $r$. How many solutions could the system have if:

2. $r = m$?

3. $r = n$?

4. $r < n$?

5. $r = m = n$?

6. $m < n$?

7. What if the linear system is homogeneous?
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