Vector Spaces

A (real) vector space $V$ is a set of things called vectors together with two operations called addition and scalar multiplication satisfying the following properties:

1. If $\vec{u}$ and $\vec{v}$ are in $V$, then $\vec{u} + \vec{v}$ is in $V$. (closed under addition)
2. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ (addition is commutative)
3. $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$ (addition is associative)
4. There is a vector called $\vec{0}$ such that $\vec{0} + \vec{u} = \vec{u}$. (additive identity)
5. For each $\vec{u}$ there is a vector $-\vec{u}$ such that $\vec{u} + (-\vec{u}) = \vec{0}$. (inverses)
6. If $\vec{u}$ is in $V$, then $c\vec{u}$ is in $V$. (closed under scalar multiplication)
7. $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$. (distributive law #1)
8. $(c + d)\vec{u} = c\vec{u} + d\vec{u}$. (distributive law #2)
9. $c(d\vec{u}) = (cd)\vec{u}$ (scalar multiplication is associative)
10. $1\vec{u} = \vec{u}$.

Since linear combinations work in vector spaces, all linear algebra works in vector spaces!
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Examples

Are these all vector spaces?

1. \(2 \times 2\) matrices, with regular addition and scalar multiplication.
2. Complex numbers, with regular addition and (real) scalar multiplication.
3. Polynomials of degree at most 2, with regular addition and scalar multiplication.
4. The set of polynomials, with regular addition and scalar multiplication.
5. Infinite sequences of real numbers, with the operations

\[
(x_1, x_2, x_3, \ldots) + (y_1, y_2, y_3, \ldots) = (x_1 + y_1, x_2 + y_2, x_3 + y_3, \ldots)
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k(x_1, x_2, x_3, \ldots) = (kx_1, kx_2, kx_3, \ldots)
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What is the (real) dimension of these vector spaces?
Are these linear transformations?

1. The function $T(a + bi) = a - bi$ from $\mathbb{C}$ to $\mathbb{C}$.
2. The function $T(\vec{v}) = ||\vec{v}||$ from $\mathbb{R}^2$ to $\mathbb{R}$.
3. The function $T \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = (a, b, c, d)$ from $2 \times 2$ matrices to $\mathbb{R}^4$.
4. The function $T(f) = f'$ from the set of polynomials to itself.
5. The function $T(f) = f'$ from the set of polynomials of degree at most 2 to itself.
6. The function $T(A) = \text{rref}(A)$ from $2 \times 2$ matrices to $2 \times 2$ matrices.
7. The function $T(p) = p(0)$ from the set of polynomials of degree at most 2 to $\mathbb{R}$. 

For those that are, what is their range, kernel, rank, and nullity? What is their associated matrix?
Vector space linear transformations

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2. The function \( T(\vec{v}) = \|\vec{v}\| \) from \( \mathbb{R}^2 \) to \( \mathbb{R} \).
3. The function \( T \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = (a, b, c, d) \) from \( 2 \times 2 \) matrices to \( \mathbb{R}^4 \).
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Amazing Vector Space Theorem!

Suppose \( V \) and \( W \) are vector spaces. An \textit{isomorphism} is a bijective linear transformation \( T : V \rightarrow W \).

\( V \) and \( W \) are \textit{isomorphic} if there exists an isomorphism between them.
Amazing Vector Space Theorem!

Suppose $V$ and $W$ are vector spaces. An *isomorphism* is a bijective linear transformation $T : V \to W$.

$V$ and $W$ are *isomorphic* if there exists an isomorphism between them.

**Theorem**

*Every* $n$-*dimensional (real) vector space is isomorphic to* $\mathbb{R}^n$. 

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