Linear Algebra
Challenge Problems 4

1. (a) Given a real number $m$, find a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that the range of $T$ is the line $y = mx$.

(b) Given real numbers $a$ and $b$, find a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ such that the range of $T$ is the plane $z = ax + by$.

2. For a constant vector $\vec{b}$ in $\mathbb{R}^3$, define $T : \mathbb{R}^3 \to \mathbb{R}^3$ by $T(\vec{x}) = \vec{x} \times \vec{b}$, i.e., the cross product of the vectors $\vec{x}$ and $\vec{b}$.

(a) Prove that $T$ is a linear transformation by finding its associated matrix.

(b) Find the range of $T$.

3. Suppose that the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ span $\mathbb{R}^n, T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, and $T(\vec{v}_i) = \vec{0}$ for all $i = 1, \ldots, p$. Prove that $T(\vec{x}) = \vec{0}$ for any vector $\vec{x} \in \mathbb{R}^n$.

4. Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation. True or false, and prove your answer:

(a) If $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ is a linearly dependent set, then $\{T(\vec{v}_1), T(\vec{v}_2), \ldots, T(\vec{v}_p)\}$ is a linearly dependent set.

(b) If $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ is a linearly independent set, then $\{T(\vec{v}_1), T(\vec{v}_2), \ldots, T(\vec{v}_p)\}$ is a linearly independent set.

(c) If $\{T(\vec{v}_1), T(\vec{v}_2), \ldots, T(\vec{v}_p)\}$ is a linearly dependent set, then $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ is a linearly dependent set.

(d) If $\{T(\vec{v}_1), T(\vec{v}_2), \ldots, T(\vec{v}_p)\}$ is a linearly independent set, then $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ is a linearly independent set.