Math 253 – Linear Algebra
Fall 2007
Team Problems #5, Due Wednesday, November 7

1. Prove that if A is a 4×4 matrix whose entries are all 1 or -1, then $\det(A)$ is divisible by 8.

2. (Section 4.7 #20a) Suppose that $B = \{\vec{b}_1, \ldots, \vec{b}_n\}$, $C = \{\vec{c}_1, \ldots, \vec{c}_n\}$, and $D = \{\vec{d}_1, \ldots, \vec{d}_n\}$ are three bases for an n-dimensional vector space. Find an equation relating the three matrices $P_{C \leftarrow B}$, $P_{D \leftarrow C}$, and $P_{D \leftarrow B}$. Justify your answer.

3. Let $\vec{v}_1$, $\vec{v}_2$, ..., $\vec{v}_{i-1}$, $\vec{v}_{i+1}$, ..., $\vec{v}_n$ be n-dimensional vectors. Define

$$T(\vec{x}) = \det[\vec{v}_1\, \vec{v}_2 \ldots \vec{v}_{i-1}\, \vec{x}\, \vec{v}_{i+1} \ldots\, \vec{v}_n].$$

In other words, $T(\vec{x})$ is the determinant of the matrix obtained by using $\vec{x}$ as the ith column. Prove that T is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}$. Does this work for any row or column?

4. Let P be a parallelogram defined by two vectors in $\mathbb{R}^2$, and let $T(\vec{x}) = A\vec{x}$ be a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$. Prove that $\det(A)$ is the ***expansion factor***

$$\frac{\text{area of } T(P)}{\text{area of } P}.$$

Using this theorem and ideas from calculus, explain why $\det(A)$ is also the expansion factor of any region in the plane.