Math 253 – Linear Algebra
Fall 2007
Team Problems #7, Due Wednesday, December 5

An ***inner product space*** is a vector space V together with an ***inner product*** $\langle \vec{x}, \vec{y} \rangle$ defined on the vectors in V, satisfying these additional four properties:

- $\langle \vec{x}, \vec{y} \rangle = \langle \vec{y}, \vec{x} \rangle$.
- $\langle \vec{x} + \vec{y}, \vec{w} \rangle = \langle \vec{x}, \vec{w} \rangle + \langle \vec{y}, \vec{w} \rangle$.
- $\langle c\vec{x}, \vec{y} \rangle = \langle \vec{x}, c\vec{y} \rangle = c\langle \vec{x}, \vec{y} \rangle$ for all $c \in \mathbb{R}$.
- $\langle \vec{x}, \vec{x} \rangle \geq 0$, and $\langle \vec{x}, \vec{x} \rangle = 0$ if and only if $\vec{x} = \vec{0}$.

Two vectors $\vec{x}$ and $\vec{y}$ are ***orthogonal*** in V if $\langle \vec{x}, \vec{y} \rangle = 0$.

1. Check that $\mathbb{R}^n$ with the usual dot product is an inner product space.
2. Check that the set of $m \times n$ matrices with inner product $\langle A, B \rangle = \text{trace}(A^T B)$ is an inner product space. Find an orthonormal basis of the set of 2×2 matrices.
3. Check that the set of continuous functions on the closed interval $[a, b]$ is an inner product space with inner product $\int_a^b f \cdot g$. Find an orthonormal basis for $\text{Span}\{1, t, t^2\}$ on the interval $[0, 1]$.
4. Show that for any vector space V and any one-to-one linear transformation $T : V \to \mathbb{R}^n$, V is an inner product space with inner product $\langle \vec{x}, \vec{y} \rangle = T(\vec{x}) \cdot T(\vec{y})$.