Classification of Subspace Intersection Graphs

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Definitions: Subspace Intersection Graphs

An e-***dimensional affine subspace*** in $\mathbb{R}^d$ is a set of points of the form

$$\{\mathbf{v}_0 + a_1\mathbf{v}_1 + \cdots + a_e\mathbf{v}_e \mid a_1, a_2, \ldots, a_e \in \mathbb{R}\}$$

for fixed vectors $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_e$ with $\mathbf{v}_1, \ldots, \mathbf{v}_e$ linearly independent.
(For example, lines in the plane, or lines or planes in 3-space.)

A graph G is a (d, e)-***subspace intersection graph*** or (d, e)-***SI graph*** if there exists a set of e-dimensional affine subspaces R in $\mathbb{R}^d$ and a one-to-one correspondence between vertices in G and subspaces in R, such that two vertices v and w in G are adjacent if and only if their corresponding subspaces intersect.

For a given graph G, if such a set of subspaces exists, it's called a (d, e)-***SI representation*** of G.

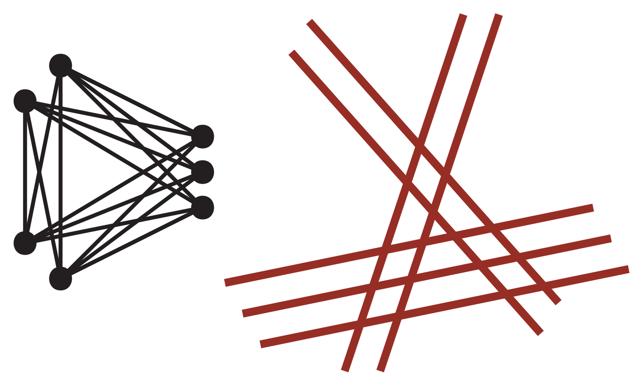
Containment Results

Theorem 5. *Every finite graph is a (d, e)-SI graph for some positive integers d and e.*

Proof sketch. Suppose G is a finite graph with vertices $v_1, \ldots, v_n$. Let $\mathcal{E} = \{e_1, \ldots, e_k\}$ be the set of all potential edges of G. We define the affine subspace V_i to be the set of all points in $\mathbb{R}^k$ with an i in coordinate p if and only if e_p is a non-edge of G incident to v_i.
We claim that $R = \{V_i\}$ is a subspace intersection representation of G in which the affine subspaces V_i might not have the same dimension. We first modify the set R by embedding $\mathbb{R}^k$ in some larger space $\mathbb{R}^j$ and using the extra dimensions to ensure that every subspace in R has the same dimension.
We now prove that R is an SI representation of G. Given two vertices v_a and v_b of G, if $v_a \sim v_b$, then for each $1 \leq p \leq k$, either V_a or V_b has points with any given value in their pth coordinate, so $V_a \cap V_b \neq \varnothing$. On the other hand, if $v_a \not\sim v_b$, and $e_p = \{v_a, v_b\}$, then points in V_a and V_b disagree in their pth coordinate, so $V_a \cap V_b = \varnothing$.
So $R = \{V_i\}$ is a (d, e)-SI representation of G, where $(d, e) \approx (n^2, n^2 - n)$. □

Theorem 6. *A graph G is a $(d, d-1)$-SI graph if and only if G is a complete multipartite graph, for all $d \geq 2$.*

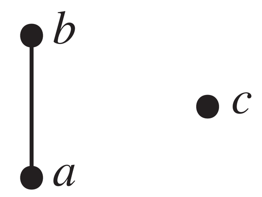
Proof sketch. Two $(d-1)$-dimensional subspaces of $\mathbb{R}^d$ are disjoint if and only if they are parallel. So a graph G has a $(d, d-1)$-SI representation if and only if it is partitionable into independent sets of vertices, with edges between every pair of vertices in different sets. These are exactly the multipartite graphs. □



Corollary. *G is a $(d, d-1)$-SI graph if and only if G is a $(2, 1)$-SI graph.* □

Separating Examples

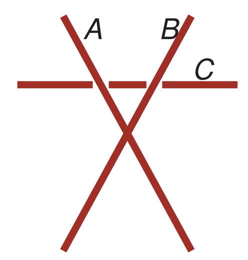
A $(3, 1)$-SI graph that's not a $(2, 1)$-SI graph



$\mathbf{G_1}$

Proposition 1. *G_1 is a $(3, 1)$-SI graph.*

Proof. Here's a $(3, 1)$-SI representation of G_1 (using lines in space).

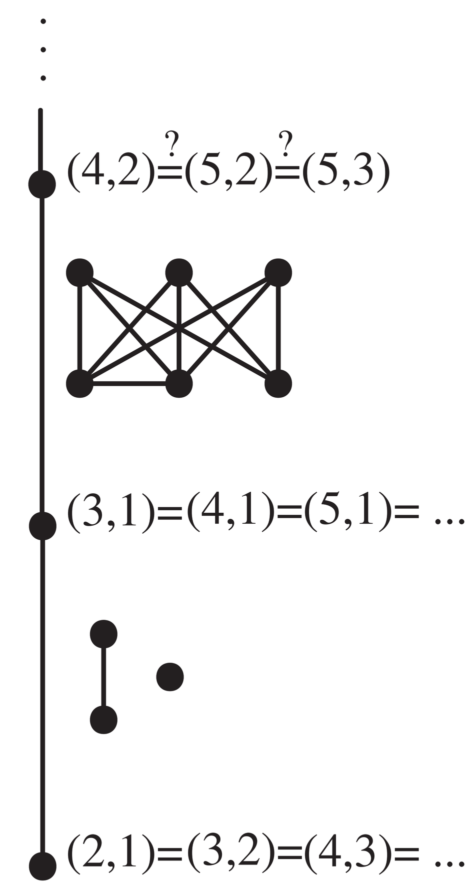


□

Proposition 2. *G_1 is not a $(2, 1)$-SI graph.*

Proof. In any $(2, 1)$-SI representation of G_1, lines A and B intersect, so C cannot be parallel to both of them. □

Hierarchy of Classes of Subspace Intersection Graphs



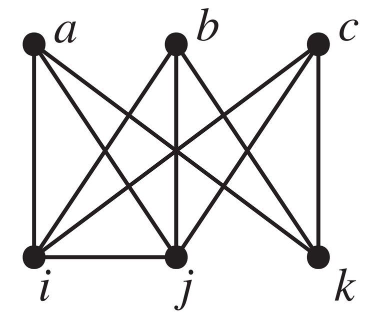
Theorem 7. *A graph G is a (d, e)-SI graph if and only if G is a $(2e + 1, e)$-SI graph, for all $d \geq 2e + 1$.*

Proof sketch. Given a set of e-spaces in $\mathbb{R}^{2e+1}$ representing G, we can easily add extra dimensions to form a set of e-spaces in $\mathbb{R}^d$ representing G.
Conversely, suppose that R is a set of e-spaces in $\mathbb{R}^d$ representing G. We consider the set $\mathcal{P}$ of all projections of R onto a $(2e+1)$-dimensional subspace of $\mathbb{R}^d$. Given two e-spaces A and B in R, almost every projection in $\mathcal{P}$ preserves their dimension and their intersection or non-intersection. Since there are only finitely many pairs of e-spaces in R, there exists a projection in $\mathcal{P}$ which preserves all pairwise intersections or non-intersections in R. □

Corollary. *For $d \geq 3$, G is a $(d, 1)$-SI graph if and only if G is a $(3, 1)$-SI graph.* □

Corollary. *For $d \geq 5$, G is a $(d, 2)$-SI graph if and only if G is a $(5, 2)$-SI graph.* □

A $(4, 2)$-SI graph that's not a $(d, 1)$-SI graph



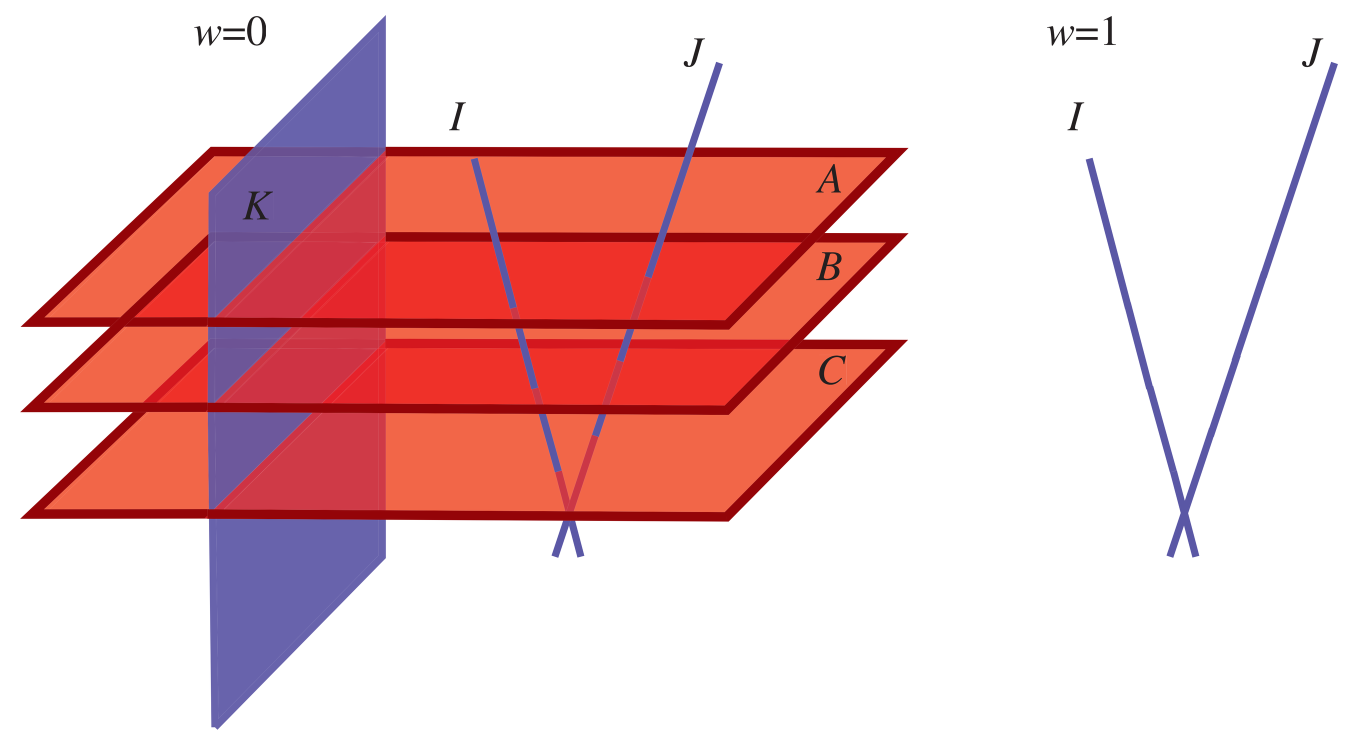
$\mathbf{G_2}$

Proposition 3. *G_2 is a $(4, 2)$-SI graph.*

Proof. Here's a $(4, 2)$-SI representation of G_2 (using planes in $\mathbb{R}^4$):

$A: z = 1, w = 0 \quad B: z = 2, w = 0 \quad C: z = 3, w = 0$
$I: x = 1, y = z \quad J: x = 1, y = -z \quad K: x = 0, w = 0$

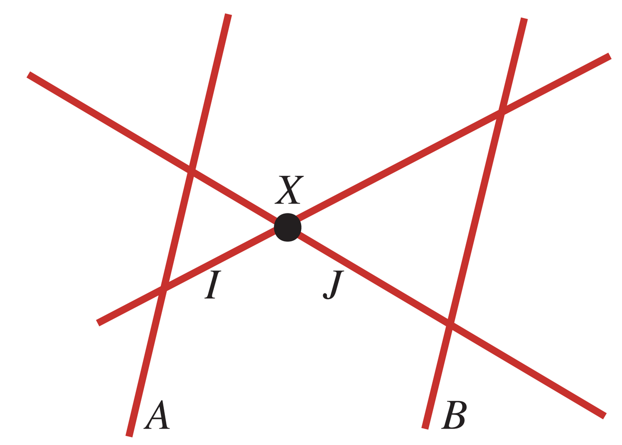
We display a picture of this representation by showing the level sets $w = 0$ and $w = 1$.



□

Proposition 4. *G_2 is not a $(d, 1)$-SI graph for any $d \geq 2$.*

Proof. Suppose by way of contradiction that $R = \{A, B, C, I, J, K\}$ is a $(d, 1)$-SI representation of G_2 with lines in $\mathbb{R}^d$. The lines I and J intersect at some point X. Since at most one of the lines A, B, and C can intersect I and J at X, the remaining two lines (say A and B) must be in the plane determined by I and J.
So A and B must be parallel. But then K must also lie in this plane. So K must intersect either I or J, which is a contradiction.



□

Open Problems

1. Find an example of a graph which is not a $(4, 2)$-SI graph.
2. Is every $(5, 2)$-SI graph a $(4, 2)$-SI graph?
3. Which graphs are intersection graphs of affine subspaces of $\mathbb{R}^3$?
4. Find a recognition algorithm for $(3, 1)$-SI graphs or other (d, e)-SI graphs.
5. **Conjecture.** *A graph G is a (d, e)-SI graph if and only if G is an $(e + 2, e)$-SI graph, for all $d \geq e + 2$.*

Acknowledgements

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