Classification of Subspace Intersection Graphs
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Definitions: Subspace Intersection Graphs
An dimensional affine subspace in \( \mathbb{R}^d \) is a set of points of the form
\[
\{ \mathbf{v}_0 + \alpha_1 \mathbf{v}_1 + \cdots + \alpha_n \mathbf{v}_n \mid \alpha_1, \ldots, \alpha_n \in \mathbb{R} \}
\]
for fixed vectors \( \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_n \) with \( \mathbf{v}_1, \ldots, \mathbf{v}_n \) linearly independent.
(For example, lines in the plane, or planes in 3-space.)

A graph \( G \) is a (d, e)-subspace intersection graph or \((d, e)-SI\) graph if there exists a set of \( e \)-dimensional affine subspaces in \( \mathbb{R}^d \) and a one-to-one correspondence between vertices in \( G \) and subspaces in \( \mathbb{R}^d \), such that two vertices \( v \) and \( w \) in \( G \) are adjacent if and only if their corresponding subspaces intersect.

For a given graph \( G \), if such a set of subspaces exists, it's called a (d, e)-SI representation of \( G \).

Hierarchy of Classes of Subspace Intersection Graphs

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Containment Results

Theorem 5. Every finite graph is a (d, e)-SI graph for some positive integers \( d \) and \( e \).

Proof sketch. Suppose \( G \) is a finite graph with vertices \( v_1, \ldots, v_n \). Let \( E = \{ (v_i, v_j) \} \) be the set of all potential edges of \( G \). We define the affine subspace \( V_i \) to be the set of all points in \( \mathbb{R}^d \) with an \( i \) in coordinate \( p \) and \( j \) in coordinate \( q \). We claim that \( R = \{ V_i \} \) is a subspace intersection representation of \( G \) in which the affine subspaces \( V_i \) might not have the same dimension. In any case, we can modify the set \( R \) by embedding \( \mathbb{R}^d \) in some larger space \( \mathbb{R}^k \) and using the extra dimensions to ensure that every subspace in \( \mathbb{R}^k \) has the same dimension.

We now prove that \( R \) is an SI representation of \( G \). Given two vertices \( v_p \) and \( v_q \) of \( G \), if \( v_p \neq v_q \), then for each \( 1 \leq i \leq n \), either \( v_i \) or \( v_j \) has points with any given value in their \( i^{th} \) coordinate, so \( V_{ij} \cap V_{jk} \neq \emptyset \). On the other hand, if \( v_p = v_q \), then every vertex in \( V_p \) and \( V_q \) is adjacent in \( \mathbb{R}^k \), so \( V_p \cap V_q = \emptyset \).

So \( R = \{ V_i \} \) is a (d, e)-SI representation of \( G \), where \( (d, e) = (r^2, \ell^2 - r^2) \).

Theorem 6. A graph \( G \) is a (d, e)-SI graph if and only if \( G \) is a complete multipartite graph, for all \( d \geq 2 \).

Proof Sketch. Two (d, e)-SI graphs are disjoint if and only if they are parallel. So a graph \( G \) has a (d, e)-SI representation if and only if it is partitionable into independent sets of vertices, with edges between every pair of vertices in different sets. These are exactly the complete multipartite graphs.

Corollary. \( G \) is a (d, e)-SI graph if and only if \( G \) is a (2, 1)-SI graph.

Theorem 7. A graph \( G \) is a (d, e)-SI graph if and only if \( G \) is a \((2e + 1, e)\)-SI graph, for all \( d \geq 2e + 1 \).

Proof sketch. Given a set of \( e \)-spaces in \( \mathbb{R}^{2e+1} \) representing \( G \), we can easily add extra dimensions to form a set of \( e \)-spaces in \( \mathbb{R}^d \) representing \( G \).

Conversely, suppose that \( R \) is a set of \( e \)-spaces in \( \mathbb{R}^d \) representing \( G \). We consider the set \( \mathbb{P} \) of all projections of \( R \) onto a \((2e + 1)\)-dimensional subspace of \( \mathbb{R}^d \). Given two \( e \)-spaces \( L \) and \( M \) in \( \mathbb{P} \), almost every projection in \( \mathbb{P} \) preserves their dimension and their intersection or non-intersection. Since there are only finitely many pairs of \( e \)-spaces in \( \mathbb{P} \), there exists a projection in \( \mathbb{P} \) which preserves all pairwise intersections or non-intersections in \( R \).

Corollary. For \( d \geq 3 \), \( G \) is a (d, d-1)-SI graph if and only if \( G \) is a (3, 1)-SI graph.

Corollary. For \( d \geq 5 \), \( G \) is a (d, d-2)-SI graph if and only if \( G \) is a (5, 2)-SI graph.

Open Problems
1. Find an example of a graph which is not a (2, 1)-SI graph.
2. Is every (3, 2)-SI graph a (4, 2)-SI graph?
3. Which graphs are intersection graphs of affine subspaces of \( \mathbb{R}^d \)?
4. Find a recognition algorithm for (3, 1)-SI graphs or other \((d, e)\)-SI graphs.
5. Conjecture. A graph \( G \) is a \((d, e)\)-SI graph if and only if \( G \) is an \((e + 2, e)\)-SI graph, for all \( d \geq e + 2 \).

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