Delta Functions

Consider a mass on a spring. We know that it is satisfies a differential equation of the form $mx'' + kx = 0$, and we give it initial conditions such as $x(0) = 0$, $x'(0) = 1$. But how did those initial conditions come to be? Physical intuitions would say that we "nudged" the mass to get it started. But how can we make mathematical since of this? In precise physical terms, to get some thing moving, we have to exert a force on it some amount of time. How do we model a "nudge" in this context? We'd like to think of it as a large force being applied for very small amount of time.

To start, think of force as the rate of change of momentum with respect to time. If p denotes momentum and $F(t)$ is a formula for the force, we have $\Delta p = p(b) - p(a) = \int_a^b F(t)dt$. To get our mass started, we are interested in changing the momentum (from 0 to 1 say). This formula shows us that any kind of force function will accomplish this goal: it is the integral of the function over a given period of time that matters, not the details of the function. So, we should model our "nudge" using a very simple force function that lasts for a very short amount of time. Something along the following lines:

$$\delta_\epsilon(t) = \begin{cases} \dfrac{1}{\epsilon} & \text{if} \quad 0 < t < \epsilon \\ 0 & \text{otherwise} \end{cases}$$

This represents a constant force of $\frac{1}{\epsilon}$ being applied for ϵ seconds, starting at $t = 0$.

Ideally, we would like to take this limit of this function as $\epsilon \to 0$. Unfortunately, such a limit does not make sense as a function. It would be defined by:

$$\delta(t) = \begin{cases} \infty & \text{if } t = 0 \\ 0 & \text{otherwise} \end{cases}$$

and we would want it to maintain the property that $\int_0^\infty \delta dt = 1$, so that it would cause our mass to start moving. It is clear that no function can have all of these properties.

Even though there is no function that has the properties that we want, the idea is quite useful, and makes sense in terms of our physical intuition. We want $\delta(t)$ to be the limit of the $\delta_\epsilon(t)$'s while that limit doesn't make sense if we look at the functions by themselves, it often does make sense if we look at the way that the $\delta_\epsilon(t)$ functions interact with other functions. To get an idea of what I'm talking about, let's look at an example. We can solve the ODE:

$$x'' + x = \delta_\epsilon(t), \quad x(0) = 0, x'(0) = 0$$

for any ϵ. (this represents a mass on a spring that is not moving when $t = 0$, when a large force is exerted on the spring for a short amount of time.) The idea is to say that the limit of those solutions as $\epsilon \to 0$, is the solution to

$$x'' + x = \delta(t)$$

To make this precise we would have to think rather hard about exactly what we mean by the limit of a set of differential equations, and we would have to make sure that the solution of such a limit is really the limit of the solutions to the original equations. These ideas were used for a long time before they could be given a rigorous mathematical foundation. At the end these notes I will give you a hint of how δ can be made to fit in a precise mathematical structure. For now we will proceed as 19th century engineers.

We use Laplace transforms to solve the equations. We take Laplace Transforms and get:

$$s^2\mathcal{L}\{x\} + \mathcal{L}\{x\} = \mathcal{L}\{\delta_\epsilon\}$$

And $\mathcal{L}\{\delta_\epsilon\}$ is:

$$\int_0^\infty \delta_\epsilon e^{-st} dt = \frac{1}{\epsilon}\int_0^\epsilon e^{-st} dt = \frac{1 - e^{\epsilon s}}{s\epsilon}$$

If we take the limit as $\epsilon \to 0$, we get $\lim_{\epsilon\to 0}\mathcal{L}\{\delta_\epsilon\} = 1$. It seems reasonable to say that in the limit, our differential equation becomes:

$$s^2\mathcal{L}\{x\} + \mathcal{L}\{x\} = 1$$

which is readily solved as

$$\mathcal{L}\{x\} = \frac{1}{s^2 + 1}$$

so

$$x = \sin(t)$$

This had exactly the effect that we wanted, the "nudge" represented by the δ functions caused the mass to start moving with an initial velocity of 1.

As we said above, the object $\delta(t)$ is not really a function, but following the lead of the example, we can use the symbol $\delta(t)$ in a differential equation when we want to represent a sudden impulse that instantaneously changes the state of the system. Even though it is not a function, we can put it into our differential equations and solve them according to the rule that $\mathcal{L}\{\delta(t)\} = 1$. This object is sometimes referred to at the Dirac δ function, and it is sometimes called the unit impulse.

We will also define

$$\delta_a(t) = \delta(t - a)$$

This represents an impulse that happens at time $t = a$ instead of $t = 0$. We can also "multiply" $\delta(t)$ by a constant b to represent an impulse that changes the system by b units instead of 1 unit. We can use the rules of Laplace transforms to get:

$$\mathcal{L}\{b\delta_a(t)\} = be^{-as}$$

It is also instructive to think of $\delta(t)$ as the derivative of the unit step function $u(t)$. This function does not change at all except at $t = 0$, where it jumps — all at once — from 0 to 1. That $\delta(t)$, is somehow the derivative of $u(t)$ is also supported by looking at the Laplace transforms: $\mathcal{L}\{u(t)\} = \frac{1}{s}$, and the derivative rule should say that $\mathcal{L}\{u'(t)\} = s\mathcal{L}\{u(t)\} - u(0)$. If we adopt the convention that $u(0) = 0$, then this agrees with $\mathcal{L}\{\delta(t)\} = 1$.

From the above example, it is not clear why we went to so much work to solve the equation when we could have just chosen the initial conditions to get what we wanted. We will now do a more complicated example that makes takes advantage of the power of the δ function. Consider the case of a drug being administered daily, we will normalize all the units to make the numbers less messy. If one dose of the drug is added to the patent, it would exponentially decay according to the differential equation:

$$x' = -rx$$

Say that each day one unit of drug is added to the system. You know that this situation can be understood by re-solving the differential equation with new initial conditions every day, but that is very tedious. Better to think of the problem in terms of impulses. The injection of new drug each day causes an almost instantaneous change in x. Injection b units of drug at time a can be represented by $x' = b\delta_a(t)$. If one unit of drug is injected every day, the differential equation becomes:

$$x' + rx = \delta(t) + \delta_1(t) + \delta_2(t) + \cdots$$

or

$$x' + rx = \sum_{n=0}^{\infty} \delta_n(t)$$

We take Laplace transforms and get:

$$\begin{aligned}
s\mathcal{L}\{x\} + r\mathcal{L}\{x\} &= \textstyle\sum_{n=0}^{\infty} \mathcal{L}\{\delta_n(t)\} \\
(s+r)\mathcal{L}\{x\} &= \textstyle\sum_{n=0}^{\infty} e^{-ns} \\
\mathcal{L}\{x\} &= \textstyle\sum_{n=0}^{\infty} \dfrac{e^{-ns}}{s+r} \\
x &= \textstyle\sum_{n=0}^{\infty} u(t-n)e^{-r(t-n)} \\
&= \begin{cases} e^{-rt} & \text{if} \quad 0 < t < 1 \\ e^{-rt} + e^{-r(t-1)} & \text{if} \quad 1 < t < 2 \\ e^{-rt} + e^{-r(t-2)} + e^{-r(t-3)} & \text{if} \quad 0 < t < 1 \\ \vdots & \end{cases}
\end{aligned}$$

It is worth some thought to check that this is the same answer that you got by doing this one step at a time in chapter 4.

Here is a way that we can look at $\delta(t)$ as a "generalized function". If you are given a function $f(t)$, you can use is to make an operator on other functions (we'll call it $\mathcal{F}$) by the following formula:

$$\mathcal{F}(g(t)) = \int_0^{\infty} f(t)g(t)dt$$

This operator takes the function $g(t)$, and returns the number $\int_0^{\infty} f(t)g(t)dt$. To make sure that this makes sense, we have to be careful which functions we're allowed to use. If we assume that $f(t) = 0$ for all t outside of a bounded set, and we make sure that $f(t)$, and $g(t)$ are bounded, then we can be sure that the integral always exists.

If we make some assumptions about the necessary integrals existing, this idea can take any function and gives us a linear operator that acts on other functions. On the other hand, there are plenty of linear operators that don't come from functions in this way. Easy examples can be defined by $\mathcal{E}_a\{f(t)\} = f(a)$. These are called evaluation operators, they consist of just evaluating the given function at $t = a$, for some fixed a.

Now we look at the $\delta_\epsilon(t)$ in this light. If $g(t)$ is continuous, then we use the mean value theorem for integrals to write

$$\int_0^{\infty} g(t)\delta_\epsilon(t)dt = \int_0^{\epsilon} g(t)\delta_\epsilon(t)dt = g(\bar{t})$$

for some $\bar{t}$ between 0 and ϵ. If we take the limit as $\epsilon \to 0$, we get $g(0)$. (Exercise 8 in section 11.5 asks you work out the details of this.) So, $\delta_a(t)$ is the "function" that corresponds to an evaluation operator. If we reexamine all that we have learned about differential equations in terms of these operators instead of in terms of functions, then $\delta(t)$ (and possibly other useful non-functions) fit right in without any difficulties. This is an idea that leads to many advanced techniques for studying differential equations.