Sample Problems For the Final

If you want more problems, you can look at the review problems at the ends of chapters 11 through 14.

1. Find equations for the following.

   (a) The line that goes through the point $(1, 2, 3)$ in the direction of $2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$.

   (b) The plane perpendicular to the line in (a) that goes through the point $(3, 2, 1)$.

   (c) A plane that passes through the points $(1, 2, 3)$, $(3, 2, 1)$, and $(6, 6, 6)$.

   (d) The tangent line to the curve defined parametrically by $x(t) = \sin(t)$, $y(t) = \sin(2t)$, and $z(t) = 1 - t^2$, at time $t = \pi$.

   (e) The tangent plane to the graph of $f(x, y) = x^2+xy+y^2$ at the point $(-1, 2, f(-1, 2))$.

   (f) The tangent plane to the level surface $f(x, y, z) = xy + yz + xz = 1$ at the point $(3, -1, 2)$.

2. The velocity of an object in space is given by $\mathbf{v}(t) = \sin(t^2 + 1)\mathbf{i} + e^t\mathbf{j} - \cos(t + 3)\mathbf{k}$. Find parametric equations for the acceleration and the position of the object if it starts at $(1, 2, 3)$ when $t = 0$.

3. Consider all of the following functions:

   $f(x, y) = x^2 + xy + y^2$, $x(t) = e^t$, $y(t) = e^{-t}$, $t(u, v) = u^2 + v^2$, $t(u, v) = u - v$.

   Calculate the following derivatives or explain why the don't make sense.

   (a) $\dfrac{\partial f}{\partial x}$ (b) $\dfrac{df}{dt}$ (c) $\dfrac{\partial f}{\partial u}$ (d) $\dfrac{\partial x}{\partial u}$ (e) $\dfrac{\partial u}{\partial x}$

4. Find the critical points of these functions and tell me if they are maximums, minimums, or saddle points.

   (a) $f(x, y) = x^2 + xy + y^2$

   (b) $f(x, y) = y^2 + x^3 - 3x$

   (c) $f(x, y) = (x - 3)(y - 2)$

   (d) $f(x, y) = x^2y^3$

5. Find the maximum and minimum values of $f(x, y) = xy$ on the circle of radius 1 centered at the origin.

6. Find the points on the circle of radius 5 centered at the origin that is closest to the point $(2, 3)$.

7. Find the maximum and minimum values of $f(x, y) = (x - 3)(y - 2)$ on the triangle with corners $(0, 0)$, $(0, 10)$, and $(10, 0)$.

8. Integrate $f(x,y) = (x-3)(y-2)$ over the triangle with corners $(0,0)$, $(0,10)$, and $(10,0)$.

9. do the following integrals:

   (a) $\iint_D x + y \, dA$, where D is the square with corners $(1,1)$, $(1,2)$, $(2,2)$, and $(2,1)$.

   (b) $\iint_D e^{x^2} \, dA$, where D is the triangle with corners $(0,0)$, $(0,2)$, $(1,2)$.

   (c) $\iint_D x + y \, dA$, where D is a circle of radius one centered at the origin.

   (d) $\iint_D e^{x^2+y^2} \, dA$, where D is a circle of radius one centered at the origin.

10. For each of the following integrals, sketch the domain and rewrite the integral with a different order of integration.

    (a) $\int_0^2 \int_{-x}^{x} x + y \, dydx$

    (b) $\int_{-1}^2 \int_x^{-x^2+2} x + y \, dydx$

11. Convert these domains to polar coordinates:

12. Do this integral by making the change of coordinates $u = x + y$, and $v = x - y$. $\iint_D (x+y)\sin(x-y) \, dA$ Where D is the square with corners $(0,0)$, $(1,1)$, $(0,2)$, and $(-1,1)$.

1. (a) $(x(t), y(t), z(t)) = (1+2t, 2-3t, 3+t)$

   (b) $2x - 3y + z = 1$

   (c) $\mathbf{A} = [6-1, 6-2, 6-3] = [5,4,3]$, and $\mathbf{B} = [6-3, 6-2, 6-1] = [3,4,5]$, $\mathbf{A} \times \mathbf{B} = [8, -16, 8]$ the plane is $8x - 16y + 8z = 0$

   (d) Point $(x(\pi), y(\pi), z(\pi)) = (0, 0, 1-\pi^2)$ tangent vector is $x'(\pi)\mathbf{i} + y'(\pi)\mathbf{j} + z'(\pi)\mathbf{k} = \mathbf{i} - 2\mathbf{j} - 2\pi\mathbf{k}$, so line is $(x,y,z) = (t, -2t, 1-\pi^2+2\pi t)$

   (e) $z = f(-1,2) + f_x(-1,2)(x+1) + f_y(-1,2)(y-2) \rightarrow z = 3 + 0(x+1) + 3(y-2) = 3 + 3(y-2)$

   (f) $\nabla f = (y+z)\mathbf{i} + (x+z)\mathbf{j} + (x+y)\mathbf{k}$, normal vector is $\nabla f(3,-1,2) = \mathbf{i} + 5\mathbf{j} + 2\mathbf{k}$ so the equation is $x + 5y + 2z = 2$

2. $\mathbf{a}(t) = 2t\cos(t^2+1)\mathbf{i} + e^t\mathbf{j} + \sin(t+3)\mathbf{k}$,

   $\mathbf{r}(t) = (\int_0^t \sin(t^2+1)\, dt + C_1)\mathbf{i} + (e^t + C_2)\mathbf{j} - (\sin(t+3) + C_3)\mathbf{k}$

   use $\mathbf{r}(0) = (1,2,3)$ to find C's.

   $\mathbf{r}(t) = (\int_0^t \sin(t^2+1)\, dt + 1)\mathbf{i} + (e^t + 1)\mathbf{j} - (\sin(t+3) + 3 - \sin(3))\mathbf{k}$

3. (a) $\dfrac{\partial f}{\partial x} = 2x + y$

   (b) $\dfrac{df}{dt} = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt}$

   (c) $\dfrac{\partial f}{\partial u} = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt}\dfrac{\partial t}{\partial u} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt}\dfrac{\partial t}{\partial u} = \ldots$

   (d) $\dfrac{\partial x}{\partial u} = \dfrac{dx}{dt}\dfrac{\partial x}{\partial u} = e^t \cdot 1 = e^{u-v}$

   (e) $\dfrac{\partial u}{\partial x}$ doesn't make sense because u does not depend of x.

4. (a) $\nabla f = (2x+y)\mathbf{i} + (x+2y)\mathbf{j} = 0$ when $2x + y = 0$ and $x + 2y = 0$, solve and get $x = 0$, and $y = 0$. $(0,0)$ is the only critical point. Use second derivatives to check for max/min/saddle: $f_{xx}f_{yy} - f_{xy}^2 = 2 \cdot 2 - 1^2 = 3$ for any (x,y). This is > 0 so it's a max or a min. Since $f_{xx} = 2$ you can see that it's a min.

   (b) $\nabla f = (3x^2 - 3)\mathbf{i} + 2y\mathbf{j} = 0$ at $(1,0)$ and $(-1,0)$. Second derivatives show that $(-1,0)$ is saddle and $(1,0)$ is min

   (c) $\nabla f = (y-2)\mathbf{i} + (x-3)\mathbf{j} = 0$ at $(3,2)$. second derivative test is no help. Note that $f > 0$ if $x < 3$ and $y > 2$ while $f < 0$ when $x > 3$ and $y < 2$. It's a saddle point because the function is both positive and negative near $(3,2)$.

   (d) $\nabla f = 2xy^3\mathbf{i} + 3x^2y^2\mathbf{j} = 0$ along x-axis and y-axis. Second derivatives give $f_{xx}f_{yy} - fxy^2 = 2y^3 \cdot 6x^2y - (6xy^2)^2$ which is 0 on the x and y-axes: second derivate is no help. Note that the function is 0 on the axes, negative in the 3rd and 4th quadrants and positive in the 1st and 2nd quadrants. So points on x-axis are saddles, points on negative y-axis are max's and points on positive y-axis are min's.

5. $f(x,y) = xy$, $g(x,y) = x^2 + y^2 = 1$. $\nabla f = \lambda \nabla g \to y = \lambda 2x, \quad x = \lambda 2y$ this tells us that $x = \pm y$, use the fact that $x^2 + y^2 = 1$ and get critical points $(\sqrt{2}/2, \sqrt{2}/2)$, $(\sqrt{2}/2, -\sqrt{2}/2)$, $(-\sqrt{2}/2, \sqrt{2}/2)$, $(-\sqrt{2}/2, -\sqrt{2}/2)$. plug them in and find that the maximum value of f is $1/2$ and the minimum is $-1/2$.

6. $f(x,y) = (x-2)^2 + (y-3)^2$, and $g(x,y) = x^2 + y^2 = 25$. Set $\nabla f = \lambda \nabla g$, $g = 25$, solve for x, y, λ and get $(10/\sqrt{13}, 15/\sqrt{13})$ as the closest point.

7. check

   - Interior critical points where $\nabla f = 0$: $(3,2)$.
   - Along bottom of triangle: $y = 0$, $f(x,0) = 6 - 2x$, no critical points.
   - Along left leg of triangle: $x = 0$, $f(0,y) = 6 - 3y$, no critical points.
   - Along hypotenuse: $y = 10 - x$, $f(x, 10-x) = (x-3)(8-x) = -x^2 + 11x - 24$, critical point at $(\frac{11}{2}, \frac{9}{2})$.
   - corners $(0,0)$, $(0,10)$, $(10,0)$.

   Check all points to find max of $\frac{25}{4}$ at $(\frac{11}{2}, \frac{9}{2})$, min of -24 at $(0,10)$

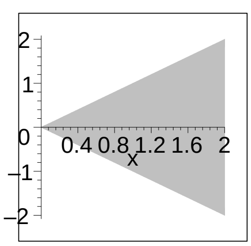
8. $\frac{350}{3}$

9. (a) 3

   (b) Oops, D was supposed to be the triangle with corners $(0,0)$, $(1,2)$, and $(1,0)$. Then the answer is $e - 1$. The answer to the problem I have written down is $-\sqrt{-\pi}\,\mathrm{erf}(\sqrt{-1}) - e + 1$.

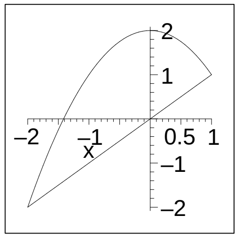
   (c) 0

   (d) $\pi e - \pi$.

10. (a) 

   $$\int_{-2}^{0} \int_{-y}^{2} x + y \; dxdy + \int_{0}^{2} \int_{y}^{2} x + y \; dxdy$$

   (b) darn, another typo. It was supposed to be $\int_{-2}^{1} \int_{x}^{2-x^2} x + y \, dydx$.

Here is a picture: 

The integral with the other limits is: $\int_{-2}^{1}\int_{-\sqrt{2-y}}^{y} x+y\,dxdy + \int_{-\sqrt{2-y}}^{\sqrt{2-y}} x+y\,dxdy$

11. (a) $r=0$ to $2\sqrt{2}$, and $\theta=0$ to $\frac{\pi}{4}$
    (b) $r=1$ to 2, and $\theta=-\frac{\pi}{2}$ to 0 (or $\frac{3\pi}{2}$ to 2π)
    (c) Must break it into two parts: $r=0$ to $2\sec(\theta)$ and $\theta=0$ to $\frac{\pi}{4}$, and $r=0$ to $2\csc(\theta)$ and $\theta=\frac{\pi}{4}$ to $\frac{\pi}{2}$

12. After change of coordinates, $(x+y)\sin(x-y)$ becomes $u\sin(v)$, and the domain becomes a square with corners $(0,0)$, $(2,0)$, $(2,-2)$, and $(0,-2)$.

    To find out what happens to $dxdy$ we have to solve for x and y and find the Jacobian. $x=\frac{u+v}{2}$ and $y=\frac{u-v}{2}$, $|J|=\frac{1}{2}$.

    the integral becomes: $\int_{-2}^{0}\int_{0}^{2} u\sin(v)\frac{1}{2}\,dudv = 2-2\cos(2)$