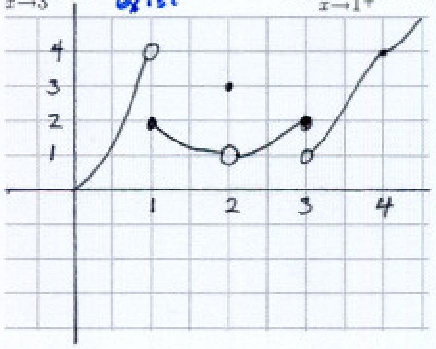
Math 141 Calculus I Fall 2004 Prof. Janeba Final Exam

Name Key Score

Instructions: This exam is closed-book and closed-notes. Anything may be stored electronically on your calculator. Consultation is allowed ONLY with the instructor. Explain and/or show work for credit.

1. For the given graph of f, find each limit, if it exists:

(a) $\lim_{x\to 3} f(x)$ doesn't exist b. $\lim_{x\to 1^+} f(x)$ c. $\lim_{x\to 4} f(x)$ d. $\lim_{x\to 2} f(x)$



a) doesn't exist as $\lim_{x\to 3^+} f(x) = 1 \neq \lim_{x\to 3^-} f(x) = 2$

b) 2

c) 4

d) 1 (not 3!)

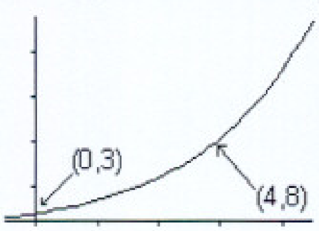
2. Find each limit exactly; if a limit fails to exist, tell what you can about the way in which it does:

(a) $\lim_{x\to 2}\frac{(x-2)^2}{x^2-4} = \lim_{x\to 2}\frac{\cancel{(x-2)}(x-2)}{\cancel{(x-2)}(x+2)} = \lim_{x\to 2}\frac{x-2}{x+2} = \frac{0}{4} = 0$

(b) $\lim_{x\to\infty}\frac{3+x-2x^2}{3x^2-5}$ [Not covered in S'09]

$$= \lim_{x\to\infty}\frac{x^2(3/x^2 + 1/x - 2)}{x^2(3 - 5/x^2)} = \lim_{x\to\infty}\frac{3/x^2 + 1/x - 2}{3 - 5/x^2} = \frac{0+0-2}{3-0} = -\frac{2}{3}$$

3. Find the exponential function $f(x) = Ca^x$ whose graph is shown:



[treated only briefly in S'09]

we know from the points on the graph that: $f(0)=3$ or $C\cdot a^0 = 3 \to C\cdot 1 = 3$ $C=3$

and $f(4)=8$ or $C\cdot a^4 = 8 \to 3\cdot a^4 = 8$

$a^4 = 8/3$

$a = \sqrt[4]{8/3} \approx 1.2779$

so $f(x) = Ca^x = 3(1.2779)^x$

4. Solve the equation ~~exactly~~ approximately ~~(don't give a decimal answer)~~ for x: $\ln(x+3) = 5$

[Hint: While we have spent almost no time on the ln function, it does have a button on our calculator]

Graph $y_1 = \ln(x+3)$, $y_2 = 5$, zoom in on the intersection point (or use a built-in intersect feature) & give the x-coordinate: $x \approx 145.413$

5. Use the definition of the derivative to find $f'(a)$ for $f(x) = 2x^2 - 5x + 1$. (Sorry, **no credit** will be given that doesn't involve the definition).

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{[2(a+h)^2 - 5(a+h) + 1] - [2a^2 - 5a + 1]}{h}$$

$$= \lim_{h\to 0} \frac{2(a^2+2ah+h^2) \cancel{-5a} - 5h \cancel{+1} - 2a^2 \cancel{+5a} \cancel{-1}}{h} = \lim_{h\to 0} \frac{\cancel{2a^2} + 4ah + 2h^2 - 5h \cancel{-2a^2}}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(4a+2h-5)}{\cancel{h}} = \lim_{h\to 0} 4a + 2h - 5 = 4a - 5$$

6. Estimate $f'(3)$ if:

(a) $f(x) = (2.3)^x$

$$f'(3) \approx \frac{f(3+.001) - f(3)}{.001} \approx 10.138$$

OR $f'(3) \approx \dfrac{f(3-.001) - f(3)}{-.001} \approx 10.130$

OR average the two

(b) f is given by the table:

x	2.7	3.0	3.3	3.6
$f(x)$	69.3	65.4	62.6	61.4

$$f'(3) \approx \frac{69.3 - 65.4}{2.7 - 3} = -13.0$$

$$f'(3) \approx \frac{62.6 - 65.4}{3.3 - 3} = -9.\overline{3}$$

avg = best est = $-11.1\overline{6}$

we can get this directly as $\dfrac{69.3 - 62.6}{2.7 - 3.3} = -11.1\overline{6}$

7. Give the equation of the line tangent to $f(x) = \dfrac{1}{x^2}$ at $a = -2$. (Use any valid method)

Slope: $f'(x) = \frac{d}{dx}(x^{-2}) = -2x^{-3} = \dfrac{-2}{x^3}$

slope at $a=-2$ $\quad f'(a) = \dfrac{-2}{a^3} = \dfrac{-2}{(-2)^3} = \dfrac{1}{4}$

point: $x = -2$, $y = f(-2) = \frac{1}{4}$

$(-2, \frac{1}{4})$

eqn of tan line: $y - y_1 = m(x - x_1)$

$y - \frac{1}{4} = \frac{1}{4}(x - -2)$

$y - \frac{1}{4} = \frac{1}{4}x + \frac{2}{4}$

$y = \frac{1}{4}x + \frac{3}{4}$

8. Find the derivative with respect to x of each expression, and simplify:

(a) $x^4 - 5x^2 + \sqrt{x} - 7$

$$\frac{d}{dx}\left(x^4 - 5x^2 + x^{1/2} - 7\right) = 4x^3 - 5 \cdot 2x + \tfrac{1}{2}x^{-1/2}$$
$$= 4x^3 - 10x + \frac{1}{2\sqrt{x}}$$

(b) $e^{5x}\left(x^2 - 7\right)$

$$\frac{d}{dx}\left[e^{5x}(x^2-7)\right] = \frac{d}{dx}\left(e^{5x}\right)(x^2-7) + e^{5x} \cdot \frac{d}{dx}\left(x^2-7\right) \quad \text{[product rule]}$$
$$= e^{5x} \cdot \frac{d}{dx}(5x)(x^2-7) + e^{5x} \cdot 2x \quad \text{[chain rule for } e^{5x}\text{]}$$
$$= 5e^{5x}(x^2-7) + e^{5x}(2x) \quad \text{[now factor out } e^{5x}\text{]}$$
$$= e^{5x}\left[5(x^2-7) + 2x\right] = e^{5x}\left[5x^2 + 2x - 35\right]$$

(c) $\cos\left(x^2\right)$

$$\frac{d}{dx}\left[\cos(x^2)\right] = -\sin(x^2) \cdot \frac{d}{dx}(x^2) \quad \text{[chain rule]}$$
$$= -2x\sin(x^2)$$

(d) $\sin\left(e^{3x}\right)$

$$\frac{d}{dx}\left[\sin(e^{3x})\right] = \cos(e^{3x}) \cdot \frac{d}{dx}(e^{3x}) \quad \text{[chain rule]}$$
$$= \cos(e^{3x}) \cdot e^{3x} \cdot \frac{d}{dx}[3x] \quad \text{[chain rule again]}$$
$$= 3e^{3x}\cos(e^{3x})$$

9. Suppose we know that $f(5) = \pi$ and $f'(5) = \frac{3}{8}$.

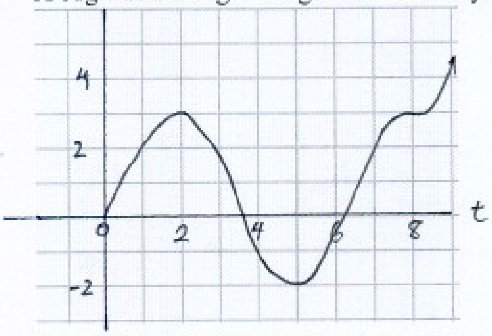
(a) If $h(x) = \ln(x) \cdot f(x)$, find $h'(5)$.

[We covered the derivative of $\ln(x)$ earlier in F'04, but all you need to know about $\ln(x)$ is that its derivative is $1/x$]

$$h'(x) = \frac{d}{dx}(\ln x) \cdot f(x) + \ln x \cdot \frac{d}{dx}(f(x))$$
$$= \tfrac{1}{x} \cdot f(x) + \ln x \cdot f'(x)$$

So
$$h'(5) = \frac{1}{5}f(5) + \ln(5) \cdot f'(5)$$
$$= \frac{1}{5} \cdot \pi + \ln(5) \cdot \frac{3}{8}$$

10. A bug walks along a straight line with varying velocity. The graph of the bug's **position** along the line as a function of time is given



(a) When is the bug's **velocity** zero? (explain)

at $t = 2, 5, 8$ – this is where the slope of the tangent line is zero, i.e. the derivative of the bug's position function. but that's velocity

(b) When is the bug's **velocity** increasing? (explain)

We are asked when is the slope on the given graph increasing, [not where is the graph's height increasing], that's where the graph is concave up: on approximately [4, 6.5] and [8, 9]

11. For the function $f(x) = x^4$: [not covered in S'09]

(a) Give the linearization for f at $a = 1$

[This is just the eqn of the tan line at $x = 1$.
If you want to practice, that's $y - 1 = 4(x-1)$
or $y = 4x - 3$]

(b) **Use your answer** in (a) to estimate $f(1.05)$.

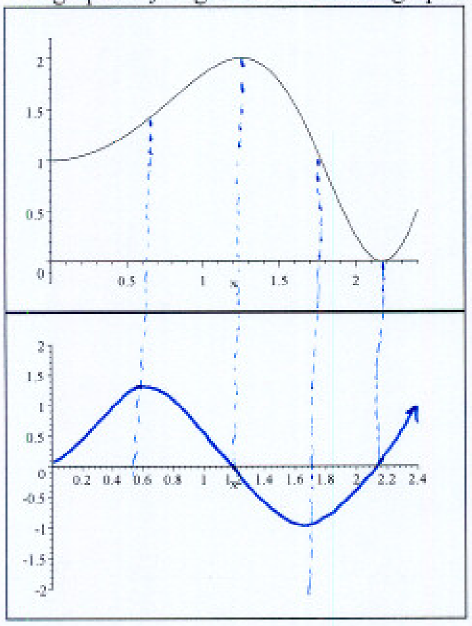
at 1.05, the tan line from (a) is at
$y = 4(1.05) - 3 = 4.2 - 3 = 1.2$
[which is approx $f(1.05) \approx 1.2155$]

12. Suppose that $W(t)$ is the amount of sand (in tons) in a certain pile, t hours after the morning shift begins at a quarry.

(a) Suppose $W'(3) = -0.38$. What does this mean, in practical terms, for the quarry manager?

3 hours after the morning shift begins, the amt of sand in the pile is decreasing at a rate of .38 tons/hour

13. The graph of f is given. Sketch the graph of $f'(x)$ on the axes provided:



remember when f has an extremum, f' changes sign.
when f has an inflection pt, f' has an extremum

14. For the function $x^2 e^{0.01x}$, find all local maxima and minima (and tell which is which):

Let $f(x) = x^2 e^{.01x}$

$$\frac{d}{dx}\left(x^2 e^{.01x}\right) = x^2 \cdot \frac{d}{dx}\left(e^{.01x}\right) + \frac{d}{dx}\left(x^2\right) e^{.01x}$$

$$= x^2 e^{.01x}(.01) + 2x e^{.01x}$$ now factor out common factor!

chain rule

$$= x e^{.01x}\left[x(.01) + 2\right] = x e^{.01x}(.01x + 2)$$

the deriv $= 0$ iff $x = 0$ OR $\underbrace{e^{.01x} = 0}_{\text{never}}$ OR $.01x + 2 = 0$

$x = 0$ OR $x = \frac{-2}{.01} = -200$

examine the graph near $x = 0$, $x = -200$

[possible windows:

$[-4,4] \times [0,5]$ to see f has a local minimum at $x = 0$

$[-300,0] \times [0, 10000]$ to see f has a local max at $x = -200$.

15. Find the global minimum of $f(x) = x^4 - 7x^2 + 16$ on $[-1, 2]$.

$$f'(x) = 4x^3 - 14x$$
$$= 2x(2x^2 - 7)$$

$f' = 0$ if $2x = 0$ or $2x^2 - 7 = 0$

$x = 0$ OR $x^2 = \frac{7}{2}$

$x = \pm\sqrt{\frac{7}{2}} = \pm 1.87$

crit pts in $[-1, 2]$ are $0, \sqrt{\frac{7}{2}}$

x	$f(x)$
-1	10
0	16
$\sqrt{\frac{7}{2}}$	3.75
2	4

global max value of 16 at $x = 0$

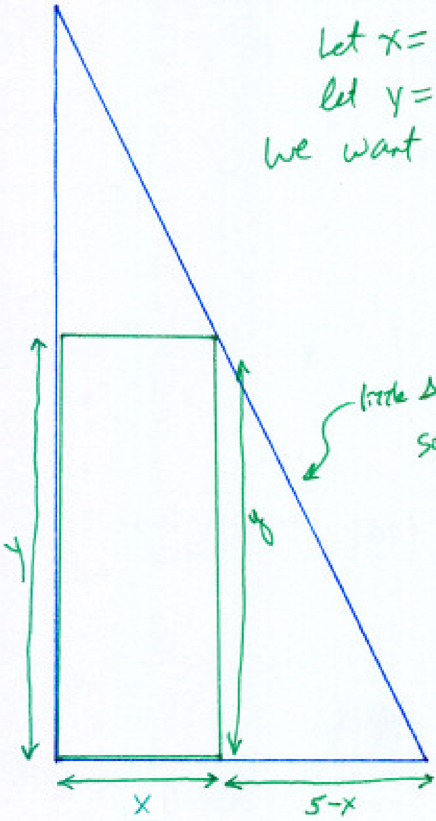
global min value of 3.75 at $x = \sqrt{\frac{7}{2}}$

16. A rectangle is to be cut from a piece of paper; the paper is in the shape of a right triangle with legs of 10 inches and 5 inches. If one corner of the rectangle is to coincide with the right angle of the triangle, how should the rectangle be made to maximize area?

Let x = width of the rectangle (along the 5" side of Δ)

let y = ht of the rectangle.

We want to maximize A = area = xy



little Δ here is similar to whole Δ.

so $\frac{y}{5-x} = \frac{10}{5}$ or $5y = 10(5-x)$

$y = 2(5-x)$

$y = 10-2x$

So we want to maximize $A = xy = x(10-2x)$ for $0 \le x \le 5$

$A = 10x - 2x^2$

$\frac{dA}{dx} = 10 - 4x$

$\frac{dA}{dx} = 0$ when $10-4x=0$

$10 = 4x$

$2.5 = x$

test the critical point:

x	$A(x)$
0	0
2.5	$2.5(10-5) = 12.5$
5	0

The maximum rectangle area A of 12.5 in^2 is obtained when the rectangle is 2.5" × 5", along the 5" and 10" sides of the Δ, respectively

17. Suppose for a function f that $f'(x) = 3\cos(x)$, and $f(0) = 4$. What is $f(x)$?

antideriv: $f(x) = 3\sin x + C$

find c: $f(0) = 3\sin(0) + C = 3 \cdot 0 + C = C$

but we're told $f(0) = 4$

$\therefore C = 4$

$f(x) = 3\sin x + 4$

18. Estimate the area under the curve e^{-x^2} from $x = 0$ to $x = 2$; describe your method.

Use the rectangle program $Y_1 = e^\wedge(-x^2)$

$A = 0$
$B = 2$
$N =$ (any large number, say 100)

$LHS \cong .8919$
$RHS = .8723$

since (looking at the graph) e^{-x^2} is monotone on $[0, 2]$,

then the area is between $.8723$ and $.8919$, or approx 0.88

19. Find each definite integral exactly:

(a) $\int_1^5 x^6 - 3x^2 + 5dx$ $= \underbrace{\frac{1}{7}x^7 - x^3 + 5x}_{\text{any antideriv}} \Big|_1^5 = \left[\frac{1}{7}(5^7) - 5^3 + 5(5)\right] - \left[\frac{1}{7} \cdot 1^7 - 1^3 + 5(1)\right]$

$= \underbrace{\frac{78125}{7} - 125 + 25}_{} - \left[\frac{1}{7} - 1 + 5\right]$

$= \frac{77425}{7} - \frac{29}{7} = \frac{77396}{7} \approx 11056.57$

(b) $\int_0^{\pi/6} \sin(x)\, dx$ Hint: remember to check your work

$= \underbrace{-\cos(x)}_{\text{antideriv}} \Big|_0^{\pi/6} = -\cos(\pi/6) - (-\cos(0))$

$= -\frac{\sqrt{3}}{2} + \cos(0) = -\frac{\sqrt{3}}{2} + 1 = \frac{2-\sqrt{3}}{2} \approx .13397$

Topics we did cover in S'09 not included here:

- Related rates
- Even & odd functions
- Graph transformations, e.g. $f(x) \to 2f(x+3)$
- Interpreting the graph of f' to tell about the graph of f. (see exam 3, S'09)

...to name a few