Solutions to Homework Assignment 17

1. (a) \((1,5\pi/2)\) and \((-1,3\pi/2)\) give the same point.
   (b) \((-2,9\pi/4)\) and \((2,5\pi/4)\) give the same point.
   (c) \((3,2+2\pi)\) and \((-3,2+\pi)\) give the same point.

3. (a) \(x = 3\cos(\pi/2) = 0, y = 3\sin(\pi/2) = 3\). The point is \((0,3)\).
   (b) \(x = 2\sqrt{2}\cos(3\pi/4) = 2\sqrt{2} \cdot -\sqrt{2}/2 = -2, y = 2\sqrt{2}\sin(3\pi/4) = 2\sqrt{2} \cdot \sqrt{2}/2 = 2\). The point is \((-2,2)\).
   (c) \(x = -\cos(\pi/3) = -1/2, y = -\sin(\pi/3) = -\sqrt{3}/2\). The point is \((-1/2,-\sqrt{3}/2)\).

5. (a) i. \(\tan \theta = 1\), so \(\theta = \pi/4\). Since \(r^2 = 1^2 + 1^2, r = \sqrt{2}\).

   ii. We may take \(r = -\sqrt{2}\) and \(\theta = \pi/4 + \pi = 5\pi/4\).

   (b) i. Here we have \(\tan \theta = -1/\sqrt{3}\), so \(\theta = 11\pi/6\) (the point is in the fourth quadrant). Since \(r^2 = 16, r = 4\).

   ii. We may take \(r = -4\) and \(\theta = 11\pi/6 - \pi = 5\pi/6\).

7. The region is the exterior of the unit circle; it is the unshaded part of the graph.

8. The region is the shaded portion of the graph; the line \(\theta = \pi/4\) is not included.

9. The region is the shaded portion of the graph.

10. The region is the shaded part.

11. The region is the shaded part.

12. The region is the shaded part.
13. Multiply both sides by $r$ to obtain $r^2 = 3r \sin \theta$, or $x^2 + y^2 = 3y$.

14. This is the vertical line $x = 1$.

15. $r^2 = 2 \sin \theta \cos \theta$, so $r^4 = (r \sin \theta)(r \cos \theta)$ (multiplying both sides by $r^2$). Thus $(x^2 + y^2)^2 = 2xy$.

16. $r + 2r \sin \theta = 1$, so $r = 1 - 2y$. Thus $r^2 = (1 - 2y)^2$, or $x^2 + y^2 = (1 - 2y)^2$.

17. $r \sin \theta = 5$.

18. $r \sin \theta = 2r \cos \theta - 1$, so $r = \frac{1}{2 \cos \theta - \sin \theta}$.

19. $r = 5$.

20. $r^2 \cos^2 \theta = 4r \sin \theta$, so $r = 4 \sec \theta \tan \theta$.

21. (a) I think it is easier in polar coordinates: $\theta = \pi/6$.
   (b) This is easier in Cartesian coordinates: $x = 3$.

22. (a) This is easier in Cartesian coordinates: $(x - 2)^2 + (y - 3)^2 = 25$.
   (b) This is easier in polar coordinates: $r = 4$.

The graphs for 23-33 odd appear below.
35. When \( \theta = 0 \), \( r = 1/2 \); when \( \theta = \pi/2 \), \( r = 2 \), etc. I put this information together to sketch the graph below.

36. When \( \theta = 0 \), \( r = 2 \); when \( \theta = \pi/2 \), \( r = 0 \). Then next time \( r = 0 \) is at \( \theta = 3\pi/2 \), and then it grows. See the graph below. Ignore the slight gap; I had to convince MAPLE to get as close as possible.

40. (a) When \( \theta = 0 \), \( r = 0 \); when \( \theta = \pi \), \( r = 1 \). Only III and VI meet these conditions. This graph will start repeating after \( 4\pi \); since (b) doesn’t repeat until after \( 8\pi \), (a) matches with VI.

(b) This one is like (a) except that \( \theta \) will ave to pass through \( 8\pi \) before repeating, so this matches with III.

(c) As \( \theta \to \pi/2 \), \( r \to \infty \), so this must correspond to IV.

(d) In this one, the values of \( r \) will both grow and return periodically to 0; it matches with V.

(e) In this one, when \( \theta = 0 \), \( r = 5 \), and when \( \theta = \pi \), \( r = -3 \) (putting the point back on the positive \( x \)-axis). This one will correspond to II.

(f) As \( \theta \) grows, \( r \to 0 \), so this corresponds to I. Notice that we are only allowed positive values of \( \theta \).

41. \[
\frac{dy}{dx} = \frac{-3 \sin \theta \sin \theta + 3 \cos \theta \cos \theta}{-3 \sin \theta \cos \theta - 3 \cos \theta \sin \theta} = -\frac{\cos 2\theta}{\sin 2\theta} = -\cot 2\theta. \text{ At } \theta = \pi/3, \text{ this is } -\frac{1/2}{\sqrt{3}/2} = \frac{1}{\sqrt{3}}.
\]

42. \[
\frac{dy}{dx} = \frac{(-\sin \theta + \cos \theta) \sin \theta + (\cos \theta + \sin \theta) \cos \theta}{(-\sin \theta + \cos \theta) \cos \theta - (\cos \theta + \sin \theta) \sin \theta}, \text{ which is } \frac{1}{-1} = -1 \text{ when } \theta = \frac{\pi}{4}.
\]

43. I did this one in class.
44. \[
\frac{dy}{dx} = \frac{\theta \sin \theta + \ln \theta \cos \theta}{\cos \theta - \ln \theta \sin \theta} = e \frac{1}{\cos \theta - \sin \theta} \quad \text{when } \theta = e.
\]

45. The tangent line is horizontal if \(\frac{dy}{d\theta} = 0\). \(y = r \sin \theta = 3 \sin \theta \cos \theta = \frac{3}{2} \sin 2\theta\). Thus \(\frac{dy}{d\theta} = 3 \cos 2\theta\), which is 0 if \(\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}, \ldots\). For these values of \(\theta\), the corresponding points are \((3/2, 3/2)\) and \((-3/2, -3/2)\). For the first point, \(r = \frac{3\sqrt{2}}{2}\); for the second, \(r = -\frac{3\sqrt{2}}{2}\).

The tangent line is vertical if \(\frac{dx}{d\theta} = 0\). \(x = r \cos \theta = 3 \cos^2 \theta\), so \(\frac{dx}{d\theta} = -6 \cos \theta \sin \theta = -3 \sin 2\theta\). This is zero if \(\theta = 0, \pi/2, \pi, 3\pi/2, 2\pi, \ldots\). The points are \((3, 0)\) and \((0, \pi/2)\).

46. \(y = e^\theta \sin \theta\), so \(\frac{dy}{d\theta} = e^\theta (\sin \theta + \cos \theta)\). This is zero only if \(\sin \theta = -\cos \theta\), or \(\tan \theta = -1\). Thus \(\theta = \ldots, -\frac{5\pi}{4}, -\frac{3\pi}{4}, -\frac{\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \ldots\). These give \(r = e^{-5\pi/4}, e^{-\pi/4}, e^{3\pi/4}, \ldots\).

\(x = r \cos \theta = e^\theta \cos \theta\). The vertical tangents appear where \(\frac{dx}{d\theta} = 0\), \(\frac{dx}{d\theta} = e^\theta (-\sin \theta + \cos \theta)\), which is 0 if \(\tan \theta = 1\). \(\tan \theta = 1\) if \(\theta = \frac{\pi}{4} + n\pi\) for some integer \(n\). The points are then \((e^{\pi+\pi n/4}, \pi+n\pi/4)\).

47. I did this one in class.

48. Rather than solving for \(r\) explicitly, I will differentiate implicitly. \(2r \frac{dr}{d\theta} = 2 \cos 2\theta\), so \(\frac{dr}{d\theta} = \frac{\cos 2\theta}{r}\). Now \(y = r \sin \theta\), so

\[
\frac{dy}{d\theta} = \frac{dr}{d\theta} \sin \theta + r \cos \theta = e^\theta \frac{\sin \theta + r \cos \theta}{r} = \frac{\cos 2\theta \sin \theta + \sin 2\theta \cos \theta}{r} = \frac{\sin 3\theta}{r}.
\]

This is zero when \(\theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \ldots\). Since \(r^2 = \sin 2\theta\), we are only permitted values of \(\theta\) that make \(\sin 2\theta\) nonnegative. The points are therefore \((0, 0), (\sqrt{3}/4, \pi/3), (\sqrt{3}/4, 4\pi/3)\).

We may similarly find the horizontal tangents; they are at \((\sqrt{3}/4, \pi/6), (\sqrt{3}/4, 7\pi/6)\), and \((0, 0)\).

51. We need to take \(\theta\) to at least \(4\pi\). The graphs for 51-54 are on the next page.